

Disturbance Decoupling of Switched Linear Systems

E. Yurtseven W.P.M.H. Heemels M.K. Camlibel

Abstract—In this paper we consider disturbance decoupling problems for switched linear systems. We will provide necessary and sufficient conditions for three different versions of disturbance decoupling, which differ based on which signals are considered to be the disturbance. In the first version the continuous exogenous input is considered as the disturbance, in the second the switching signal and in the third both of them are considered as disturbances. The solutions of the three disturbance decoupling problems will be based on geometric control theory for switched linear systems and will entail both mode-dependent and mode-independent static state feedback.

Index Terms—Disturbance decoupling, switched linear systems, invariant subspaces, geometric control theory

I. INTRODUCTION

Geometric control theory for linear time-invariant systems has a long and rich history, as is evidenced by the availability of various textbooks on the topic [2, 6, 18]. In particular, for solving disturbance decoupling problems (DDPs) for linear systems the usage of geometric theory turned out to be extremely powerful. Also solutions to various problems for smooth nonlinear systems using a nonlinear geometric approach are available in the literature, see, e.g., [3, 7, 11]. However, outside the context of linear or smooth nonlinear control systems, the number of results on DDPs is rather limited. This is specifically surprising for hybrid dynamical systems or subclasses such as switched systems [9], as they have been studied extensively over the last two decades.

Only a few results are available on geometric control theory and solutions to DDPs for switched systems. In [8] the largest controlled invariant set for switched linear systems (SLSs) is studied in which both the switching (discrete control input) and the continuous input can be manipulated as control inputs. In the context of linear parameter-varying (LPV) systems various parameter-varying (controlled and conditioned) invariant subspaces are introduced in [1] and various algorithms are presented to compute them. Based on [1] first results in the direction of applying these concepts to DDPs with respect to continuous disturbances are given in [14] using parameter-dependent state feedback. Recently, in [12] the DDP for switched linear systems using mode-dependent state feedback control is solved and combined with results on quadratic stabilizability [4, 17]. Also for reachability problems for SLSs invariant subspaces played an important role. In particular, in [15, 16] it was shown

that the reachable set of an SLS is equal to the smallest controlled invariant set containing the subspace spanned by all the input matrices of the individual subsystems. For switched nonlinear systems, the only work known to the authors is [20]. In [20] local versions of DDP with respect to continuous disturbances are solved using both mode-dependent and mode-independent static state feedback.

The objectives of this paper are to provide complete answers to DDPs for SLSs using various new controlled and conditioned invariant subspaces for SLSs. We first assume that the control input is absent and analyze the disturbance decoupling properties of a SLS. In contrast with the above mentioned references, which only study disturbance decoupling (DD) with respect to continuous exogenous disturbances, we consider three variants of DD as will be formally defined in Section II, namely DD with the disturbances being either (i) the exogenous continuous disturbances, (ii) the switching signal, or (iii) both the continuous disturbances and the switching signal. In Section III we will fully characterize these three DD properties. In Section IV we will add continuous control inputs to the problem and solve the DDP using state feedback controllers that may be both mode-dependent and mode-independent. We will allow for direct feedthrough terms of the control input into the to-be-decoupled output variable, a situation that was not considered in the aforementioned references. Note also that variant (ii) and (iii) are in the present paper for the first time. In Section V we provide algorithms to compute the largest *common* controlled invariant subspaces using both mode-dependent and mode-independent feedback, which can be used to verify the characterizations of the solvability of the DDPs provided in Section IV and also to construct feedbacks solving the DDPs. For space reasons we omit all proofs, which can found in [19].

II. PROBLEM FORMULATION

A switched linear system (SLS) without control inputs is described by the following equations

$$\dot{x}(t) = A_{\sigma(t)}x(t) + E_{\sigma(t)}d(t) \quad (1a)$$

$$z(t) = H_{\sigma(t)}x(t) \quad (1b)$$

where $x(t) \in \mathbb{R}^{n_x}$, $d(t) \in \mathbb{R}^{n_d}$ and $z(t) \in \mathbb{R}^{n_z}$ denote the state variable, the exogenous input and output, respectively, at time $t \in \mathbb{R}_+ := [0, \infty)$. For each $i \in \{1, \dots, M\}$, $A_i \in \mathbb{R}^{n_x \times n_x}$, $E_i \in \mathbb{R}^{n_x \times n_d}$ and $H_i \in \mathbb{R}^{n_z \times n_x}$ are matrices describing a linear subsystem. Switching between subsystems (modes) is orchestrated by the switching signal σ . We assume that σ lies in the set \mathcal{S} of right-continuous

Evren Yurtseven and Maurice Heemels are with the Eindhoven University of Technology, Dept. of Mechanical Engineering, Hybrid and Networked Systems group, The Netherlands, e.yurtseven@student.tue.nl, m.heemels@tue.nl.

Kanat Camlibel is with the University of Groningen, Inst. of Mathematics and Computer Science, Groningen, The Netherlands m.k.camlibel@rug.nl.

functions $\mathbb{R}_+ \rightarrow \{1, \dots, M\}$ that are piecewise constant with a finite number of discontinuities in a finite length interval. Particular switching signals are the constant ones $\sigma^i \in \mathcal{S}, i = 1, \dots, M$, which are defined as $\sigma^i(t) = i$ for all $t \in \mathbb{R}_+$. We assume that the exogenous signal d is locally integrable, i.e. $d \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_d})$. Clearly, the SLS (1) has for each $d \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_d})$, initial condition $x(0) = x_0 \in \mathbb{R}^{n_x}$ and switching signal $\sigma \in \mathcal{S}$ a unique solution $x_{x_0, \sigma, d}$ and a corresponding output $z_{x_0, \sigma, d}$.

We will consider the following three variants of disturbance decoupling, which differ based on which signals are considered to be the disturbance.

Definition II.1 *The SLS (1) is called disturbance decoupled (DD) with respect to d if*

$$z_{x_0, \sigma, d_1} = z_{x_0, \sigma, d_2} \quad (2)$$

for all $x_0 \in \mathbb{R}^{n_x}, \sigma \in \mathcal{S}$ and $d_1, d_2 \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_d})$.

Definition II.2 *The SLS (1) is called disturbance decoupled (DD) with respect to σ if*

$$z_{x_0, \sigma_1, d} = z_{x_0, \sigma_2, d} \quad (3)$$

for all $x_0 \in \mathbb{R}^{n_x}, \sigma_1, \sigma_2 \in \mathcal{S}$ and $d \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_d})$.

Definition II.3 *The SLS (1) is called disturbance decoupled (DD) with respect to both σ and d if*

$$z_{x_0, \sigma_1, d_1} = z_{x_0, \sigma_2, d_2} \quad (4)$$

for all $x_0 \in \mathbb{R}^{n_x}, \sigma_1, \sigma_2 \in \mathcal{S}$ and $d_1, d_2 \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_d})$.

Remark II.4 *DD with respect to continuous disturbances d is commonly studied and well motivated within the context of linear systems [2, 6, 18] and nonlinear systems [3, 7, 11]. DD with respect to σ or both σ and d is typical for switched systems as studied here. These variants of DD are relevant in situations where the switching signal σ is uncontrolled and we would like to design a (closed-loop) system in which σ does not influence certain important performance variables z . In particular, when σ models certain faults in the system such as breakage of pipes, actuators, sensors, etc. and thus each mode $i \in \{1, \dots, M\}$ corresponds to one of these discrete fault scenarios, it would be desirable to decouple z from σ (and possibly other continuous disturbances d). Hence, as such these variants of DD constitute fundamental problems in the area of fault-tolerant control [10]. Another motivation for DD with respect to both σ and d are DD problems for piecewise linear systems, which is discussed in detail in [19].*

III. DISTURBANCE DECOUPLING CHARACTERIZATIONS

To provide characterizations for the above mentioned DD properties, we need to introduce some concepts and a technical lemma. We call a subspace $\mathcal{V} \in \mathbb{R}^{n_x}$ A -invariant for $A \in \mathbb{R}^{n_x \times n_x}$, if $A\mathcal{V} \subseteq \mathcal{V}$. We call a subspace $\{A_1, \dots, A_M\}$ -invariant $A_i \in \mathbb{R}^{n_x \times n_x}, i = 1, \dots, M$, if $A_i\mathcal{V} \subseteq \mathcal{V}$ for all $i = 1, \dots, M$. Given a matrix $A \in \mathbb{R}^{n_x \times n_x}$

and a subspace $\mathcal{W} \in \mathbb{R}^{n_x}$, let $\langle A|\mathcal{W} \rangle$ denote the smallest A -invariant subspace that contains \mathcal{W} , i.e.,

$$\langle A|\mathcal{W} \rangle = \mathcal{W} + A\mathcal{W} + \dots + A^{n_x-1}\mathcal{W} \quad (5)$$

For a set of matrices $\{A_1, \dots, A_M\}$ and a subspace \mathcal{W} , the smallest $\{A_1, \dots, A_M\}$ -invariant subspace that contains \mathcal{W} , denoted by $\mathcal{V}_s(\mathcal{W})$, is uniquely defined by the following three properties:

- 1) $\mathcal{W} \subseteq \mathcal{V}_s(\mathcal{W})$;
- 2) $\mathcal{V}_s(\mathcal{W})$ is $\{A_1, \dots, A_M\}$ -invariant;
- 3) For any subspace \mathcal{V} being $\{A_1, \dots, A_M\}$ -invariant with $\mathcal{W} \subseteq \mathcal{V}$, it holds that $\mathcal{V}_s(\mathcal{W}) \subseteq \mathcal{V}$.

Calculation of $\mathcal{V}_s(\mathcal{W})$ can be done using the recurrence relation

$$\mathcal{V}_1 = \mathcal{W}; \quad \mathcal{V}_{i+1} = \sum_{j=1}^M \langle A_j|\mathcal{V}_i \rangle$$

Since $\mathcal{V}_i \subseteq \mathcal{V}_{i+1}$ for $i = 1, 2, \dots$ and $\mathcal{V}_p = \mathcal{V}_{p+1}$ implies $\mathcal{V}_q = \mathcal{V}_p$ for all $q \geq p$, it holds that $\mathcal{V}_q = \mathcal{V}_s(\mathcal{W})$ for all $q \geq n_x$, see, e.g., [1, 16].

The reachable set of (1) is defined as $\mathcal{R} := \{x_{0, \sigma, d}(T) \mid T \in \mathbb{R}_+, \sigma \in \mathcal{S} \text{ and } d \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_d})\}$ being the set of states that can be reached from the origin in finite time for some σ and d .

Lemma III.1 *For the SLS (1),*

$$\mathcal{R} = \mathcal{V}_s\left(\sum_{i=1}^M \text{im } E_i\right)$$

See [15, 16] for the proof of this lemma.

A. Disturbance decoupling with respect to d

In this section we consider DD with respect to d . Before giving the main result of this subsection, we would like to give a motivating example which shows that this is not a trivial problem.

Example III.2 *Consider a bimodal SLS as in (1) with the first subsystem described as*

$$\dot{x}_1(t) = 0 \quad \dot{x}_2(t) = d(t) \quad z(t) = x_1(t) \quad (6)$$

and the second subsystem as

$$\dot{x}_1(t) = d(t) \quad \dot{x}_2(t) = 0 \quad z(t) = x_2(t) \quad (7)$$

It is obvious that both the linear subsystems are DD with respect to d . However, under the switching signal $\sigma(t)$ described as

$$\sigma(t) = \begin{cases} 1 & 0 \leq t < t_1 \\ 2 & t_1 \leq t \end{cases} \quad (8)$$

the output at t_1 is given by

$$z(t_1) = x_{20} + \int_0^{t_1} d(\tau) d\tau$$

showing that the SLS is not DD with respect to d . Therefore, this example shows that it is not sufficient that the subsystems

of an SLS are DD with respect to d for the SLS itself to be DD with respect to d .

The observation in the above example is consistent with the following theorem.

Theorem III.3 *The SLS (1) is DD with respect to d if and only if an $\{A_1, \dots, A_M\}$ -invariant subspace \mathcal{V} exists such that*

$$\sum_{i=1}^M \text{im } E_i \subseteq \mathcal{V} \subseteq \ker \begin{bmatrix} H_1 \\ \vdots \\ H_M \end{bmatrix} \quad (9)$$

Example III.4 *Reconsidering the bimodal SLS in Example III.2, one can see that*

$$\sum_{i=1}^2 \text{im } E_i = \mathbb{R}^2 \quad \ker \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = \{0\}$$

Clearly, there cannot be a subspace \mathcal{V} that satisfies

$$\sum_{i=1}^2 \text{im } E_i \subseteq \mathcal{V} \subseteq \ker \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$$

Therefore, the SLS in Example III.2 is not DD with respect to d according to Theorem III.3, which agrees with our previous observation based on computing the output of the SLS explicitly.

B. Disturbance decoupling with respect to σ

In this section we consider DD with respect to σ in which we will use \mathcal{N}_i being the unobservable subspace corresponding to the pair (H_i, A_i) , i.e., $\mathcal{N}_i = \ker H_i \cap \ker H_i A_i \cap \dots \cap \ker H_i A_i^{n_x-1}$. Note that \mathcal{N}_i is also the largest A_i -invariant subspace contained in $\ker H_i$, $i \in \{1, \dots, M\}$.

Theorem III.5 *The SLS (1) is DD with respect to σ if and only if for all $(i, j) \in \{1, \dots, M\} \times \{1, \dots, M\}$ the following conditions hold*

- (i) $H_i = H_j$;
- (ii) $\mathcal{N}_i = \mathcal{N}_j$;
- (iii) $\text{im}(A_i - A_j) \subseteq \mathcal{N}_i$;
- (iv) $\text{im}(E_i - E_j) \subseteq \mathcal{N}_i$.

Based on Theorem III.5, we can also derive an alternative characterization of DD with respect to σ , which is more geometric in nature.

Corollary III.6 *The SLS (1) is DD with respect to σ if and only if there exists an $\{A_1, \dots, A_M\}$ -invariant subspace \mathcal{V} such that for all $(i, j) \in \{1, \dots, M\} \times \{1, \dots, M\}$*

- (i) $H_i = H_j$
- (ii) $\text{im}(A_i - A_j) \subseteq \mathcal{V} \subseteq \ker H_i$
- (iii) $\text{im}(E_i - E_j) \subseteq \mathcal{V}$

C. Disturbance decoupling with respect to σ and d

Using the above results, we will characterize DD with respect to σ and d now.

Lemma III.7 *The SLS (1) is DD with respect to both σ and d if and only if it is DD with respect to σ and DD with respect to d .*

Theorem III.8 *The following statements are equivalent:*

- 1) *The SLS (1) is DD with respect to σ and d .*
- 2) *The conditions*
 - (i) $H_i = H_j = H$,
 - (ii) $\mathcal{N}_i = \mathcal{N}_j = \mathcal{N}$,
 - (iii) $\text{im}(A_i - A_j) \subseteq \mathcal{N}$, and
 - (iv) $\text{im} E_i \subseteq \mathcal{N}$*hold for all $(i, j) \in \{1, \dots, M\} \times \{1, \dots, M\}$.*
- 3) *There exists an $\{A_1, \dots, A_M\}$ -invariant subspace \mathcal{V} such that*
 - (i) $H_i = H_j = H$,
 - (ii) $\text{im}(A_i - A_j) \subseteq \mathcal{V} \subseteq \ker H$, and
 - (iii) $\text{im} E_i \subseteq \mathcal{V}$*for all $(i, j) \in \{1, \dots, M\} \times \{1, \dots, M\}$.*

IV. DDP BY STATE FEEDBACK

In the previous section we provided full characterizations of DD properties. Now we will consider if and how we should choose control inputs in order to render a SLS disturbance decoupled in some sense. In order to do so, consider the SLS

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + E_{\sigma(t)}d(t) \quad (10a)$$

$$z(t) = H_{\sigma(t)}x(t) + J_{\sigma(t)}u(t) \quad (10b)$$

where we included now a control input $u(t) \in \mathbb{R}^{n_u}$ at time $t \in \mathbb{R}_+$. As before we denote the solution corresponding to $x_0 \in \mathbb{R}^{n_x}$, $\sigma \in \mathcal{S}$, $d \in \mathcal{L}_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_d})$ and $u \in \mathcal{L}_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_u})$ by $x_{x_0, \sigma, d, u}$ and the corresponding output by $z_{x_0, \sigma, d, u}$. We are now interested in finding conditions under which controllers can be found such that the closed-loop system is DD with respect to d , to σ , or to both. We start with static state feedback controllers.

A. Solution of DDP with respect to d by mode-dependent state feedback

Problem IV.1 *The disturbance decoupling problem with respect to d (DDPd) by mode-dependent state feedback for SLS (10) amounts to finding $F_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \dots, M$ such that*

$$\dot{x}(t) = (A_{\sigma(t)} + B_{\sigma(t)}F_{\sigma(t)})x(t) + E_{\sigma(t)}d(t) \quad (11a)$$

$$z(t) = (H_{\sigma(t)} + J_{\sigma(t)}F_{\sigma(t)})x(t) \quad (11b)$$

is DD with respect to d .

Note that the SLS (11) results from putting the system (10) in closed loop with $u(t) = F_{\sigma(t)}x(t)$, which requires knowledge of the active mode $\sigma(t)$ at time $t \in \mathbb{R}_+$.

Definition IV.2 Consider the SLS (10) with $d = 0$. A subspace \mathcal{V} is called output-nulling $\{(A_1, B_1), \dots, (A_M, B_M)\}$ -invariant if for any $x_0 \in \mathcal{V}$ and $\sigma \in \mathcal{S}$ there exists a control input $u \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_u})$ such that $x_{x_0, \sigma, 0, u}(t) \in \mathcal{V}$ and $z_{x_0, \sigma, 0, u}(t) = 0$ for all $t \in \mathbb{R}_+$.

Sometimes an output-nulling $\{(A_1, B_1), \dots, (A_M, B_M)\}$ -invariant subspace is called a common output-nulling controlled invariant subspace for (10).

Definition IV.3 Given a linear subspace $\mathcal{V} \subseteq \mathbb{R}^{n_x}$, we define the extended subspace $e(\mathcal{V}) \subseteq \mathbb{R}^{n_x + n_z}$ as

$$e(\mathcal{V}) = \mathcal{V} \times \{0\}$$

Theorem IV.4 Consider the SLS (10) with $d = 0$. Let \mathcal{V} be a subspace of \mathbb{R}^{n_x} . The following statements are equivalent.

- (i) \mathcal{V} is common output-nulling controlled invariant.
- (ii) $\begin{bmatrix} A_j \\ H_j \end{bmatrix} \mathcal{V} \subseteq e(\mathcal{V}) + \text{im} \begin{bmatrix} B_j \\ J_j \end{bmatrix}$ for all $j = 1, \dots, M$.
- (iii) There exist $F_j \in \mathbb{R}^{n_u \times n_x}$, $j = 1, \dots, M$, such that $(A_j + B_j F_j) \mathcal{V} \subseteq \mathcal{V} \subseteq \ker(H_j + J_j F_j)$ for all $j = 1, \dots, M$.

Let $\{\mathcal{V}_j \mid j \in \mathcal{J}\}$ be a collection of common output-nulling controlled invariant subspaces for the SLS (10). It follows from Definition IV.2 that $\sum_{j \in \mathcal{J}} \mathcal{V}_j$ is common output-nulling controlled invariant. Therefore, the set of all common output-nulling controlled invariant subspaces admits a largest element. The largest common output-nulling controlled invariant subspace for a given SLS plays a crucial role in the solution of DDPd by mode-dependent feedback.

Definition IV.5 Consider the SLS (10) with $d = 0$. We define \mathcal{V}_{md}^* as the largest common output-nulling controlled invariant subspace for the SLS (10) that is

- (i) \mathcal{V}_{md}^* is common output-nulling controlled invariant;
- (ii) if \mathcal{V} is a common output-nulling controlled invariant subspace for the SLS (10), then $\mathcal{V} \subseteq \mathcal{V}_{md}^*$.

Theorem IV.6 Consider the SLS (10). DDPd by mode-dependent feedback is solvable if and only if

$$\sum_{i=1}^M \text{im} E_i \subseteq \mathcal{V}_{md}^* \quad (12)$$

In case (12) is satisfied a mode-dependent feedback $u(t) = F_{\sigma(t)} x(t)$ that renders (11) DD with respect to d is characterized by $(A_i + B_i F_i) \mathcal{V}_{md}^* \subseteq \mathcal{V}_{md}^* \subseteq \ker(H_i + J_i F_i)$, $i \in \{1, \dots, M\}$.

In Section V we will provide an algorithm to compute \mathcal{V}_{md}^* for a given SLS.

Remark IV.7 For the special case that $J_i = 0$, $i = 1, \dots, M$, this problem was solved also in [12]. In [12] the DDP with respect to d by mode-dependent state feedback was combined with the question of quadratic stability. Sufficient conditions were given exploiting known results for quadratic

stabilization as in [4, 17]. These stability conditions can also be added to the theorems that we present here, but unfortunately, they are, just as in [12], not so trivial to verify, certainly for a high number of subsystems. Indeed, the sufficient conditions that guarantee solvability of DDP with quadratic stability (DDPQS) according to Theorem 3.2 in [12] are the existence of F_j , $j = 1, \dots, M$, such that $(A_j + B_j F_j) \mathcal{V}_{md}^* \subseteq \mathcal{V}_{md}^* \subseteq \ker(H_j + J_j F_j)$ for all $j = 1, \dots, M$ and there exists a convex combination $\sum \alpha_j (A_j + B_j F_j)$ with $\sum_{j=1}^M \alpha_j = 1$ and $\alpha_j \geq 0$, $j = 1, \dots, M$, being a Hurwitz matrix. Since a parameterization of all F_j , $j = 1, \dots, M$, satisfying $(A_j + B_j F_j) \mathcal{V}_{md}^* \subseteq \mathcal{V}_{md}^* \subseteq \ker(H_j + J_j F_j)$ is hard to come by in the first place, and one has to search for both α_j and F_j , $j = 1, \dots, M$, which is a non-convex problem, these conditions are not easy to verify. Also for the satisfaction of the stabilization condition it is unclear if using \mathcal{V}_{md}^* instead of another common output-nulling controlled invariant subspace containing $\sum_{i=1}^M \text{im} E_i$ is introducing conservatism into the conditions. Another difference with respect to [12], is that DDP with respect to d by mode-independent state feedback was not considered in [12], while we treat this problem in the next section.

B. Solution of DDP with respect to d by mode-independent state feedback

Problem IV.8 The disturbance decoupling problem with respect to d (DDPd) by mode-independent feedback for SLS (10) amounts to finding $F \in \mathbb{R}^{n_u \times n_x}$ such that

$$\dot{x}(t) = (A_{\sigma(t)} + B_{\sigma(t)} F)x(t) + E_{\sigma(t)} d(t) \quad (13a)$$

$$z(t) = (H_{\sigma(t)} + J_{\sigma(t)} F)x(t) \quad (13b)$$

is DD with respect to d .

Note that the SLS (13) results from putting the system (10) in closed loop with $u(t) = Fx(t)$. The latter state feedback controller does not require knowledge of the active mode $\sigma(t)$ at time $t \in \mathbb{R}_+$.

Definition IV.9 Consider the SLS (10) with $d = 0$. A subspace \mathcal{V} is called output-nulling $\{(A_1, B_1), \dots, (A_M, B_M)\}$ -invariant under mode-independent control if for any $x_0 \in \mathcal{V}$ there exists a control input $u \in L_1^{loc}(\mathbb{R}_+, \mathbb{R}^{n_u})$ such that $x_{x_0, \sigma, 0, u}(t) \in \mathcal{V}$ and $z_{x_0, \sigma, 0, u}(t) = 0$ for all $\sigma \in \mathcal{S}$ and for all $t \in \mathbb{R}_+$.

Sometimes a subspace that is output-nulling $\{(A_1, B_1), \dots, (A_M, B_M)\}$ -invariant under mode-independent control is called a common output-nulling controlled invariant subspace under mode-independent control for (10).

Theorem IV.10 Consider the SLS (10) with $d = 0$. Let \mathcal{V} be a subspace of \mathbb{R}^{n_x} . Define the matrices $A_s \in \mathbb{R}^{M(n_x + n_z) \times n_x}$

and $B_s \in \mathbb{R}^{M(n_x+n_z) \times n_u}$

$$A_s = \begin{bmatrix} A_1 \\ H_1 \\ \vdots \\ A_M \\ H_M \end{bmatrix}, B_s = \begin{bmatrix} B_1 \\ J_1 \\ \vdots \\ B_M \\ J_M \end{bmatrix} \quad (14)$$

and $e(\mathcal{V})^M$ as $e(\mathcal{V})^M = \overbrace{e(\mathcal{V}) \times e(\mathcal{V}) \times \dots \times e(\mathcal{V})}^{M \text{ times}}$. The following statements are equivalent.

- (i) \mathcal{V} is common output-nulling controlled invariant under mode-independent control.
- (ii) $A_s \mathcal{V} \subseteq e(\mathcal{V})^M + \text{im} B_s$.
- (iii) There exists $F \in \mathbb{R}^{n_u \times n_x}$ such that $(A_j + B_j F) \mathcal{V} \subseteq \mathcal{V} \subseteq \ker(H_j + J_j F)$ for all $j = 1, \dots, M$.

Definition IV.11 Consider the SLS (10) with $d = 0$. We define \mathcal{V}_{mi}^* as the largest common output-nulling controlled invariant subspace under mode-independent control for the SLS (10), that is

- 1) \mathcal{V}_{mi}^* is common output-nulling controlled invariant under mode-independent control;
- 2) if \mathcal{V} is common output-nulling controlled invariant under mode-independent control for the SLS (10), then $\mathcal{V} \subseteq \mathcal{V}_{mi}^*$.

Theorem IV.12 Consider the SLS (10). DDPd by mode-independent feedback is solvable if and only if

$$\sum_{i=1}^M \text{im } E_i \subseteq \mathcal{V}_{mi}^* \quad (15)$$

In case (15) is satisfied a mode-independent feedback $u(t) = Fx(t)$ that renders (13) DD with respect to d is characterized by $(A_i + B_i F) \mathcal{V}_{mi}^* \subseteq \mathcal{V}_{mi}^* \subseteq \ker(H_i + J_i F)$, $i \in \{1, \dots, M\}$.

In Section V we will present an algorithm to compute the largest common output-nulling controlled invariant subspace using mode-independent feedback for a given SLS.

C. Solution of DDP with respect to σ by mode-dependent state feedback

Consider the SLS (10).

Problem IV.13 The disturbance decoupling problem with respect to σ (DDP σ) by mode-dependent state feedback amounts to finding $F_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \dots, M$ such that the SLS (11) is DD with respect to σ .

Before giving the theorem for the solvability of Problem IV.13 we need to introduce the following lemma.

Lemma IV.14 Consider the SLS (10). DDP σ by mode-dependent feedback is solvable if and only if there exist a common output-nulling controlled invariant subspace \mathcal{V} and $F_k \in \mathbb{R}^{n_u \times n_x}$, $k = 1, \dots, M$, with $(A_k + B_k F_k) \mathcal{V} \subseteq$

$\mathcal{V} \subseteq \ker(H_k + J_k F_k)$, $k = 1, \dots, M$, such that the following three conditions hold for all $(i, j) \in \{1, \dots, M\} \times \{1, \dots, M\}$:

- (i) $H_i + J_i F_i = H_j + J_j F_j$,
- (ii) $\text{im}(A_i + B_i F_i - A_j - B_j F_j) \subseteq \mathcal{V}$,
- (iii) $\text{im}(E_i - E_j) \subseteq \mathcal{V}$.

Theorem IV.15 Consider the SLS (10). DDP σ by mode-dependent feedback is solvable if and only if there exist $G_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \dots, M$, such that for all $(i, j) \in \{1, \dots, M\} \times \{1, \dots, M\}$ it holds that

- (i) $H_i + J_i G_i = H_j + J_j G_j$,
- (ii) $\text{im}(A_i + B_i G_i - A_j - B_j G_j) \subseteq \mathcal{V}_{md}^*$,
- (iii) $\text{im}(E_i - E_j) \subseteq \mathcal{V}_{md}^*$.

In case the above conditions are satisfied a mode-dependent feedback $u(t) = F_{\sigma(t)} x(t)$ that renders (11) DD with respect to σ can be constructed by letting $\{v_1, v_2, \dots, v_{n_x}\}$ be a basis for \mathbb{R}^{n_x} such that $\{v_1, \dots, v_q\}$ is a basis for \mathcal{V}_{md}^* and defining

$$F_i v_k = \begin{cases} \tilde{F}_i v_k & k \in \{1, 2, \dots, q\} \\ G_i v_k & k \in \{q+1, q+2, \dots, n_x\} \end{cases}$$

in which \tilde{F}_i satisfies $(A_i + B_i \tilde{F}_i) \mathcal{V}_{md}^* \subseteq \mathcal{V}_{md}^* \subseteq \ker(H_i + J_i \tilde{F}_i)$, $i \in \{1, \dots, M\}$.

D. Solution of DDP with respect to d and σ (DDP $d\sigma$) by mode-dependent state feedback

Consider the SLS (10).

Problem IV.16 The disturbance decoupling problem with respect to d and σ (DDP $d\sigma$) by mode-dependent state feedback amounts to finding $F_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \dots, M$ such that the SLS (11) is DD with respect to d and σ .

Theorem IV.17 Consider the SLS (10). DDP $d\sigma$ by mode-dependent feedback is solvable if and only if there exist $G_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, \dots, M$ such that for all $(i, j) \in \{1, \dots, M\} \times \{1, \dots, M\}$ it holds that

- (i) $H_i + J_i G_i = H_j + J_j G_j$,
- (ii) $\text{im}(A_i + B_i G_i - A_j - B_j G_j) \subseteq \mathcal{V}_{md}^*$,
- (iii) $\text{im } E_i \subseteq \mathcal{V}_{md}^*$.

In case the above conditions are satisfied a mode-dependent feedback $u(t) = F_{\sigma(t)} x(t)$ that renders (11) DD with respect to d and σ can be constructed by letting $\{v_1, v_2, \dots, v_{n_x}\}$ be a basis for \mathbb{R}^{n_x} such that $\{v_1, \dots, v_q\}$ is a basis for \mathcal{V}_{md}^* and defining

$$F_i v_k = \begin{cases} \tilde{F}_i v_k & k \in \{1, 2, \dots, q\} \\ G_i v_k & k \in \{q+1, q+2, \dots, n_x\} \end{cases}$$

in which \tilde{F}_i satisfies $(A_i + B_i \tilde{F}_i) \mathcal{V}_{md}^* \subseteq \mathcal{V}_{md}^* \subseteq \ker(H_i + J_i \tilde{F}_i)$, $i \in \{1, \dots, M\}$.

Remark IV.18 In Theorem IV.15 and Theorem IV.17, if \mathcal{V}_{md}^* is replaced with \mathcal{V}_{mi}^* and $G_i = G_j$ for all $(i, j) \in \{1, \dots, M\} \times \{1, \dots, M\}$, then the characterizations of solvability of DDP σ and DDP $d\sigma$ by mode-independent

feedback, respectively, are obtained. The construction of the disturbance decoupling feedbacks have to be adapted accordingly, see [19] for more details.

V. ALGORITHMS TO TEST THE HYPOTHESES OF THE SOLUTIONS TO THE DDP

A. Algorithm for Theorem IV.6

In this subsection we will present an algorithm to find the largest common output-nulling controlled invariant subspace for the SLS (10).

Algorithm V.1

$$\mathcal{V}_0 = \mathbb{R}^{n_x}; \quad (16a)$$

$$\mathcal{V}_{i+1} = \bigcap_{j=1}^M \{x \mid \exists u \ A_j x + B_j u \in \mathcal{V}_i, \ H_j x + J_j u = 0\} \quad (16b)$$

From this recurrence relation it follows that $\mathcal{V}_{i+1} \subseteq \mathcal{V}_i$ for all $i = 0, 1, \dots$, and if $\mathcal{V}_k = \mathcal{V}_{k+1}$ for some k , then $\mathcal{V}_i = \mathcal{V}_k$ for all $i \geq k$. Let q be the smallest $k \in \mathbb{N}$ such that $\mathcal{V}_k = \mathcal{V}_{k+1}$. Obviously, $q \leq n_x$. We claim that $\mathcal{V}_q = \mathcal{V}_{md}^*$.

Theorem V.2 Consider the SLS (10) and Algorithm V.1. Then, $\mathcal{V}_q = \mathcal{V}_{md}^*$ with $q := \min\{k \in \mathbb{N} \mid \mathcal{V}_k = \mathcal{V}_{k+1}\} \leq n_x$.

B. Algorithm for Theorem IV.12

In this subsection we will present the algorithm to find the largest common output-nulling controlled invariant under mode-independent control for the SLS (10).

Algorithm V.3 Define the matrices A_s and B_s in the same way as in Theorem IV.10.

$$\mathcal{V}_0 = \mathbb{R}^{n_x}; \quad \mathcal{V}_{i+1} = \{x \mid A_s x \in \epsilon(\mathcal{V}_i)^M + \text{im } B_s\} \quad (17)$$

As above, it holds that $\mathcal{V}_{i+1} \subseteq \mathcal{V}_i$ for all $i = 0, 1, \dots$ and if $\mathcal{V}_k = \mathcal{V}_{k+1}$ for some k , then $\mathcal{V}_i = \mathcal{V}_k$ for all $i \geq k$. Let q be the smallest $k \in \mathbb{N}$ such that $\mathcal{V}_k = \mathcal{V}_{k+1}$. Obviously, $q \leq n_x$. We claim that $\mathcal{V}_q = \mathcal{V}_{mi}^*$.

Theorem V.4 Consider the SLS (10) and Algorithm V.3. Then, $\mathcal{V}_q = \mathcal{V}_{mi}^*$ with $q := \min\{k \in \mathbb{N} \mid \mathcal{V}_k = \mathcal{V}_{k+1}\} \leq n_x$.

Remark V.5 The algorithm of Section V-A to obtain the largest common controlled invariant subspace (with mode-dependent feedback) for a SLS inside another subspace is related to the algorithm presented before in [1, 14] for LPV systems for the special case that $J_i = 0$, $i = 1, \dots, M$. The algorithm in Section V-B for mode-independent feedback was not presented in the literature before.

Remark V.6 A mode-dependent measurement feedback based solution of DDPd is provided in [19].

VI. CONCLUSIONS

In this paper three different disturbance decoupling (DD) properties for switched linear systems were analyzed. The difference between the three properties is induced by which signals are considered as the disturbances: (i) the continuous exogenous signal, (ii) the switching signal, or (iii) both the continuous exogenous signal and the switching signal. Complete geometric characterizations for these properties were given, which were used to solve also disturbance decoupling problems (DDPs) by suitable choice of controllers. Both mode-dependent and mode-independent static state feedback controllers were considered for the three instances of the DDP. We used common controlled and common conditioned invariant subspaces to characterize these properties. Algorithms to compute these subspaces were provided as well, so that these results can be applied by straightforward computations.

REFERENCES

- [1] G. Balas, J. Bokor, and Z. Szabó. Invariant Subspaces for LPV Systems and Their Application. *IEEE Transactions on Automatic Control*, 48:2065–2069, 2003.
- [2] G.B. Basile and G. Marro. *Controlled and Conditioned Invariants in Linear System Theory*. Prentice Hall, Englewood Cliffs, NJ, 1992.
- [3] C.I. Byrnes. Feedback decoupling of rotational distributions for spherically constrained systems. In *Proc. 23rd IEEE Conf. on Decision and Control*, pages 421–426, Las Vegas, Nevada, USA, 1984.
- [4] E. Feron. Quadratic stabilizability of switched systems via state and output feedback. Ciscp-468, MIT Technical Report, 1996.
- [5] A.F. Filippov. *Differential Equations with Discontinuous Righthand Sides*. Mathematics and its Applications. Kluwer, The Netherlands, 1988.
- [6] H.L. Trentelman and A.A. Stoorvogel and M. Hautus. *Control Theory for Linear Systems*. Communications and control engineering series. Springer, 2002.
- [7] A. Isidori, A.J. Krener, C. Gori-Giorgi, and S. Nonaco. Nonlinear decoupling via feedback: a different geometric approach. 26(2):331–345, *IEEE Trans. Autom. Control*.
- [8] A.A. Julius and A.J. van der Schaft. The Maximal Controlled Invariant Set of Switched Linear Systems. In *Proc. of the 41st IEEE Conf. on Decision and Control*, pages 3174–3179, Las Vegas, Nevada USA, 2002.
- [9] D. Liberzon. *Switching in Systems and Control*. Systems and Control: Foundations and Applications. Birkhauser, Boston, MA, 2003.
- [10] J. Lunze and J. H. Richter. Reconfigurable fault-tolerant control: A tutorial introduction. *European Journal of Control*, 14(5):359–386, 2008.
- [11] H. Nijmeijer and A.J. Van der Schaft. Controlled invariance for nonlinear systems. *IEEE Trans. Autom. Control*, 27(4):904–914, 1982.
- [12] N. Otsuka. Disturbance decoupling with quadratic stability for switched linear systems. *Systems and Control Letters*, 2010, to appear.
- [13] J.M. Schumacher. Compensator synthesis using (C, A, B) -pairs. *IEEE Trans. Aut. Contr.*, 25:1133–1138, 1980.
- [14] G. Stikkel, J. Bokor, and Z. Szabó. Disturbance Decoupling Problem with Stability for LPV Systems. In *European Control Conference*, Cambridge, U.K., 2003.
- [15] Zhendong Sun and S.S. Ge. *Switched Linear Systems: Control and Design*. Communications and control engineering series. Springer, 2005.
- [16] Zhendong Sun, S.S. Ge, and T.H. Lee. Controllability and Reachability Criteria for Switched Linear Systems. *Automatica*, 38:775–786, 2002.
- [17] M. Wicks, P. Peleties, and R. DeCarlo. Switched controller synthesis for the quadratic stabilization of a pair of unstable linear systems. *Eur. J. Contr.*, 4:140–147, 1998.
- [18] W.M. Wonham. *Linear multivariable control: A geometric approach*. Springer, 3rd edition, 1985.
- [19] E. Yurtseven, W.P.M.H. Heemels, and M.K. Camlibel. Disturbance decoupling of switched linear systems. *submitted for journal publication*.
- [20] L. Zhang, D. Cheng, and C. Li. Disturbance decoupling of switched nonlinear systems. *IEE Proceedings*, 152(1):49–54, 2005.