An embedding approach for the design of state-feedback tracking controllers for references with jumps

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SUMMARY

We study the problem of designing state-feedback controllers to track time-varying state trajectories that may exhibit jumps. Both plants and controllers considered are modeled as hybrid dynamical systems, which are systems with both continuous and discrete dynamics, given in terms of a flow set, a flow map, a jump set, and a jump map. Using recently developed tools for the study of stability in hybrid systems, we recast the tracking problem as the task of asymptotically stabilizing a set, the *tracking set*, and derive conditions for the design of state-feedback tracking controllers with the property that the jump times of the plant coincide with those of the given reference trajectories. The resulting tracking controllers guarantee that solutions of the plant starting close to the reference trajectory stay close to it and that the difference between each solution of the controlled plant and the reference trajectory converges to zero asymptotically. Constructive conditions for tracking control design in terms of LMIs are proposed for a class of hybrid systems with linear maps and input-triggered jumps. The results are illustrated by various examples. Copyright © 2013 John Wiley & Sons, Ltd.

1. INTRODUCTION

The literature on stability analysis and stabilization of equilibria for systems with state jumps is relatively well developed; see, for example, [1–5] for results for hybrid systems, [6–8] for results in the scope of measure differential inclusions, [9, 10] in the scope of complementarity systems, [11, 12] in the scope of impulsive dynamical systems, and many others. On the other hand, in many control problems, such as tracking, output regulation, synchronization, and observer design, the goal consists of stabilizing *time-varying* trajectories. To effectively tackle such problems for hybrid systems, results on the stability and stabilization of time-varying trajectories of such systems are imperative. Unfortunately, general results for stabilizing impulsive/discontinuous or, more generally, hybrid trajectories are not currently available. Notable specific solutions to stabilization of such trajectories are given by the work in [13–16], in which the state estimation and tracking problems for particular classes of mechanical systems with impacts are addressed, the work in [17], in which an observer-based control design problem for a class of complementarity systems is studied, the work in [18], in which tracking control and observer design problems for a class of measure differential inclusions is solved, the work in [19–22], in which a tracking problem for a class of mechanical systems

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with unilateral constraints is addressed, and the work in [23] considering the juggling problem as a tracking problem.

In this paper, we present sufficient conditions characterizing controllers solving a state tracking control problem for a general class of hybrid systems. More specifically, we consider plants given in terms of a constrained flow equation[‡]

$$\dot{\xi} = f_p(\xi, u), \qquad (\xi, u) \in C_p \tag{1}$$

and a constrained jump inclusion

$$\xi^+ \in G_p(\xi, u), \qquad (\xi, u) \in D_p, \tag{2}$$

with output function given by the identity, that is, $y = \xi$. The set C_p is the *flow set*, which is where continuous evolution or *flows* are possible. The single-valued map f_p, called the *flow map*, defines the differential equation governing the flows. The set D_p is the *jump set*, which collects the points from where discrete evolution or *jumps* are possible. From this set, the state is updated via the set-valued map G_p, called the *jump map*. For this class of hybrid systems, a controller assigning the input u and measuring ξ is to be designed such that the difference between ξ and the reference trajectory r, which may both flow and jump, is well behaved in a sense to be made more precise later. Without being precise about a notion of tracking at this point, it should be expected that the tracking controller guarantees both stability and attractivity properties relative to the reference trajectory. Stability consists of the property that solutions to the plant starting close to the reference stay close to it, whereas attractivity consists of the property that the distance between the plant's solution component and the reference decreases asymptotically. A major challenge in guaranteeing these properties for hybrid systems is that the jump times of the reference trajectory and the state of the controlled plant do not coincide in general. In Section 2, we discuss in more detail how this complicates the tracking control design problem. The proposed approach in this article consists of recasting the state tracking problem for hybrid systems, which is defined in Section 3, as the stabilization of a closed set that embeds the reference trajectory. Exploiting sufficient conditions for asymptotic stability of closed sets for hybrid systems, in Section 4, we present sufficient conditions for a class of hybrid state-feedback tracking controllers enforcing that the jump times of the plant coincide with those of the given reference trajectory. Section 5 considers a special case of hybrid systems with linear maps and jumps triggered by the inputs and proposes constructive conditions for control design given in terms of LMIs. The proposed approach and results are illustrated by examples in Section 6. Section 7 presents concluding remarks.

Notation. The following notational conventions are used throughout the paper. The n-dimensional Euclidean space is denoted by $\mathbb{R}^n$, real numbers by $\mathbb{R}$, nonnegative real numbers by $\mathbb{R}_{\geqslant 0}$, and natural numbers including 0 by $\mathbb{N}$. Given a set S, $\overline{S}$ denotes its closure; given a vector $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean vector norm; given a set $S \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, we define $|x|_S := \inf_{y \in S} |x - y|$ as the distance from x to the set S. Given vectors x and y, we write $[x^\top y^\top]^\top$ with the shorthand notation (x, y). The inner product operation between two vectors x and y is denoted $\langle x, y \rangle$. Given a continuously differentiable function V, ∇V denotes its gradient. Given a locally Lipschitz function V, $V^\circ(x, w)$ denotes the Clarke-generalized derivative of V at x in the direction w [24], that is, $V^\circ(x, w) = \max_{\zeta \in \partial V(x)} \langle \zeta, w \rangle$, where $\partial V(x)$ is the generalized gradient of V in the sense of Clarke, which is a closed, convex, and nonempty set equal to the convex hull of all limit sequences of $\nabla V(x_i)$ with $x_i \to x$ taking value away from every set of measure zero in which V is nondifferentiable. A function $\alpha : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is said to belong to class-$\mathcal{K}$ ($\alpha \in \mathcal{K}_\infty$) if it is continuous, zero at zero, and strictly increasing and to belong to class-$\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if it belongs to class-$\mathcal{K}$ and is unbounded. $\mathcal{PD}$ denotes the set of real-valued positive definite functions, that is, $\rho \in \mathcal{PD}$ implies $\rho(0) = 0$ and $\rho(s) > 0$ for all $s \neq 0$. The identity function is denoted Id. For a symmetric matrix A, we denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ its minimum and maximum eigenvalues, respectively.

[‡] As defined in Section 3, the solutions to (1) and (2) will be given by (absolutely) continuous functions on each interval of flow.

2. OBSTACLES TO TRACKING CONTROL DESIGN FOR HYBRID SYSTEMS

To illustrate the difficulties in tracking control design problems for systems with state jumps, consider a scalar, single-valued hybrid plant as in (1) and (2) with state ξ and output $y = \xi$, and the reference trajectory to be tracked given by the sawtooth signal shown in Figure 1, which has discontinuities when reaching 1. Trajectories ξ to the plant can be specified as functions defined on hybrid time domains dom ξ, which are subsets of $\mathbb{R}_{\geq 0} \times \mathbb{N}$, and parameterize the trajectories by flow time t and jump time j, see [5]. A common approach used in tracking control of continuous-time and discrete-time plants consists of defining the tracking error and then analyzing the resulting time-varying error dynamics; a different approach to tracking control that is based on dynamic inversion appeared in [25]. Following the former approach, the reference trajectory r on the hybrid time domain dom r is given by

$$r(t, j) = t - t_j^r \qquad \forall t \in \left[t_j^r, t_{j+1}^r\right], \tag{3}$$

where $t_j^r = j$, $j \in \mathbb{N}$. Note that $r(t, j) \in [0, 1]$ for all $(t, j) \in$ dom r, where dom r is the union of $[t_j^r, t_{j+1}^r] \times \{j\}$ for every $j \in \mathbb{N}$. Let

$$\mathcal{T}_r := \bigcup_{j \in \mathbb{N}_{>0}} \left\{(t_j^r, j - 1)\right\} \tag{4}$$

denote the fixed values of (t, j) at which r jumps. Then, the dynamics of the tracking error e defined as

$$e = y - r(t, j) = \xi - r(t, j)$$

are given by the flow equation

$$\dot{e} = f_p(e + r(t, j), u) - 1 \tag{5}$$

when

$$(e + r(t, j), u) \in C_p \qquad \text{and} \qquad t \in \left[t_j^r, t_{j+1}^r\right], \tag{6}$$

and by the jump equation

$$e^+ = G_e(e + r(t, j), u, t, j) \tag{7}$$

when

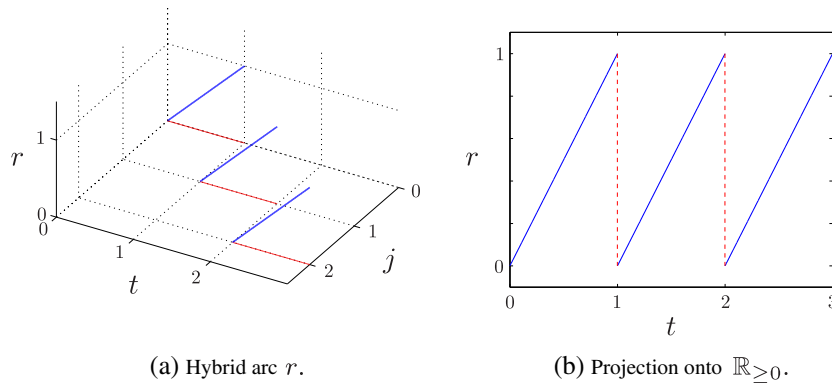$$(e + r(t, j), u) \in D_p \qquad \text{or} \qquad (t, j) \in \mathcal{T}_r, \tag{8}$$

(a) Hybrid arc r.   (b) Projection onto $\mathbb{R}_{\geq 0}$.

Figure 1. Reference trajectory for the tracking control problem in Section 2.
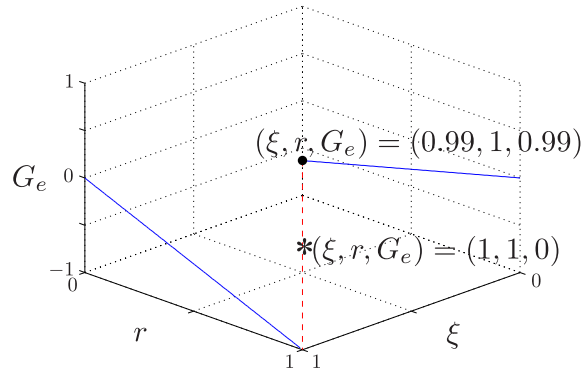
Figure 2. A resulting jump map G_e for the error system in the tracking control problem of Section 2 when $G_p(\xi, u) = 0$ and $D_p = \{(\xi, u) : \xi = 1\}$. The map G_e is defined for each $(\xi, r) \in (\mathbb{R} \times \{1\}) \times (\{1\} \times [0, 1])$.

where G_e is defined at every point satisfying (8) as

$$G_e(e + r, u, t, j) = \begin{cases} G_p(e + r, u) - r & (e + r, u) \in D_p, (t, j) \notin \mathcal{T}_r \\ e + r & (e + r, u) \notin D_p, (t, j) \in \mathcal{T}_r \\ G_p(e + r, u) & (e + r, u) \in D_p, (t, j) \in \mathcal{T}_r \end{cases},$$

where G_p is the jump map of plants (1) and (2).

The first challenge to tracking control for hybrid systems is due to the impossibility of writing conditions (6) and (8) in terms of the tracking error solely. Note that this issue also emerges in the continuous-time and discrete-time settings, but it is aggravated in hybrid systems because the flow and jump conditions depend on the state. As a consequence, the dynamics of the tracking error depend on ξ and r. Now, suppose that a feedback law $u = \kappa_c(y, r)$ is designed to map the error to zero when both the plant's state ξ and r jump simultaneously, that is, the third case in the definition of G_e yields $G_e = 0$. The second challenge to tracking control emerges from the possibility that, from points ξ in C_p that are nearby D_p and times $(t, j) \in \mathcal{T}_r$, G_e updates e to $|e^+| = e + 1$, which is far from zero (also when e is arbitrarily close to zero). In fact, Figure 2 depicts a particular map G_e as a function of ξ and $r(t, j)$ when the jumps of the plant occur when $\xi = 1$, that is, $D_p := \{(\xi, u) : \xi = 1\}$, and with $G_p(\xi, u) = \xi + u$, $\kappa_c(y, r) = -\xi$, resulting in $G_p(\xi, u) = 0$ when $\xi = 1$. In this case, when $(t, j) \in \mathcal{T}_r$ (equivalently, $r(t, j) = 1$), if $\xi = 1$, then $e^+ = 0$ (i.e., the tracking error jumps to zero); however, if ξ is slightly below 1 and $(t, j) \in \mathcal{T}_r$, then $|e^+|$ will be close to 1 after the jump (of the reference). This *peaking phenomenon*, which is due to the jump instants of plant and reference not coinciding, has also been recognized in [8, 13, 16, 17, 21] and imposes a difficulty in guaranteeing that the norm of e converges to zero.

We consider tracking controllers that tackle the first challenge earlier while avoiding the issue of an increasing error signal in the second challenge by ensuring that jumps of the plant occur at the same instant as the jumps of the reference trajectories. For this purpose, we recast the tracking control problem as the stabilization of a closed set that embeds the time-varying reference trajectory. For the design of the tracking controllers, we exploit sufficient conditions for asymptotic stability of hybrid systems in [26] (see also [27] and [4]). An alternative approach for tracking control for hybrid systems based on generating the reference trajectories from an exosystem was proposed in [28].

3. PROBLEM STATEMENT

In this paper, we consider the state tracking problem for plants modeled as hybrid systems (1) and (2). In this way, a plant is denoted $\mathcal{H}_p$ and has state $\xi \in \mathbb{R}^{n_p}$, input $u \in \mathbb{R}^{m_p}$, and output $y = \xi$. Its

dynamics are given by

$$\mathcal{H}_p \begin{cases} \dot{\xi} &= f_p(\xi, u) & (\xi, u) \in C_p \\ \xi^+ &\in G_p(\xi, u) & (\xi, u) \in D_p \\ y &= h(\xi) := \xi \end{cases} . \tag{9}$$

The set $C_p \subset \mathbb{R}^{n_p} \times \mathbb{R}^{m_p}$ is the *flow set*, the function $f_p : C \to \mathbb{R}^{n_p}$ is the *flow map*, the set $D_p \subset \mathbb{R}^{n_p} \times \mathbb{R}^{m_p}$ is the *jump set*, $G_p : D \rightrightarrows \mathbb{R}^{n_p}$ is the *jump map*,[§] and h is the output map, given by the identity function. The data of the hybrid system $\mathcal{H}_p$ are thus given by $(C_p, f_p, D_p, G_p, \mathrm{Id})$.

Solutions to hybrid systems $\mathcal{H}_p$ are defined by pairs of hybrid arcs ϕ_p and hybrid inputs u on hybrid time domains. Hybrid time domains are subsets of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ given by the union of intervals of the form $[t_j, t_{j+1}] \times \{j\}$, $t_j \leq t_{j+1}$. More precisely, a set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a compact hybrid time domain if

$$E = \bigcup_{j=0}^{J-1} \left([t_j, t_{j+1}] \times \{j\} \right)$$

for some finite sequence of jump instants $0 = t_0 \leq t_1 \leq t_2 \cdots \leq t_J$. It is a hybrid time domain if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, \ldots J\})$ is a compact hybrid time domain. A function $\phi_p : \mathrm{dom}\, \phi_p \to \mathbb{R}^{n_p}$ is a hybrid arc if $\mathrm{dom}\, \phi_p$ is a hybrid time domain and, for each $j \in \mathbb{N}$, the function $t \mapsto \phi_p(t, j)$ is absolutely continuous on the interval $\{t : (t, j) \in \mathrm{dom}\, \phi_p\}$. A function $u : \mathrm{dom}\, u \to \mathbb{R}^{m_p}$ is a hybrid input if $\mathrm{dom}\, u$ is a hybrid time domain and, for each $j \in \mathbb{N}$, the function $t \mapsto u(t, j)$ is Lebesgue measurable and locally essentially bounded on the interval $\{t : (t, j) \in \mathrm{dom}\, u\}$. Then, a hybrid arc $\phi_p : \mathrm{dom}\, \phi_p \to \mathbb{R}^{n_p}$ and a hybrid input $u : \mathrm{dom}\, u \to \mathbb{R}^{m_p}$ define a *solution pair* (ϕ_p, u) *to the hybrid system* $\mathcal{H}_p$ *from the initial condition* $\phi_p(0, 0)$ if the following conditions hold:

(S0) $(\phi_p(0, 0), u(0, 0)) \in \overline{C}_p \cup D_p$, and $\mathrm{dom}\, \phi_p = \mathrm{dom}\, u \ (= \mathrm{dom}\,(\phi_p, u))$.

(S1) For each $j \in \mathbb{N}$ such that $I_j := \{t : (t, j) \in \mathrm{dom}\,(\phi_p, u)\}$ has nonempty interior $\mathrm{int}(I_j)$,

$$(\phi_p(t, j), u(t, j)) \in C_p \text{ for all } t \in \mathrm{int}(I_j),$$

and, for almost all $t \in I_j$,

$$\frac{d\phi_p}{dt}(t, j) = f_p(\phi_p(t, j), u(t, j)).$$

(S2) For each $(t, j) \in \mathrm{dom}\,(\phi_p, u)$ such that $(t, j+1) \in \mathrm{dom}\,(\phi_p, u)$,

$$(\phi_p(t, j), u(t, j)) \in D_p, \qquad \phi_p(t, j+1) \in G_p(\phi_p(t, j), u(t, j)).$$

A solution pair (ϕ_p, u) to $\mathcal{H}_p$ is said to be *complete* if $\mathrm{dom}\,(\phi_p, u)$ is unbounded, *Zeno* if it is complete but the projection of $\mathrm{dom}\,(\phi_p, u)$ onto $\mathbb{R}_{\geq 0}$ is bounded, *discrete* if its domain is $\{0\} \times \mathbb{N}$, and *maximal* if there does not exist another pair $(\phi_p, u)'$ such that (ϕ_p, u) is a truncation of $(\phi_p, u)'$ to some proper subset of $\mathrm{dom}\,(\phi_p, u)'$.

We consider hybrid arcs $r : \mathrm{dom}\, r \to \mathbb{R}^{n_p}$ defining reference trajectories to be tracked by the plant $\mathcal{H}_p$. As for the construction of hybrid time domains earlier, the sequence of times corresponding to the jump instants of a reference trajectory r is denoted $0 = t_0^r \leq t_1^r \leq t_2^r \leq \cdots$. By using the definition of $\mathcal{H}_p$ in (9), hybrid tracking controllers with state $\eta \in \mathbb{R}^{n_c}$ and data $(C_c, f_c, D_c, G_c, \kappa_c)$ are given by

$$\mathcal{H}_c \begin{cases} \dot{\eta} &= f_c(\eta, y, r) & (\eta, y, r) \in C_c \\ \eta^+ &\in G_c(\eta, y, r) & (\eta, y, r) \in D_c \\ u &= \kappa_c(\eta, y, r) \end{cases} . \tag{10}$$

The input of $\mathcal{H}_c$ has been assigned to (y, r), whereas its output u to the input of the plant $\mathcal{H}_p$.

[§]The notation $\rightrightarrows$ indicates that G_p is a set-valued map, that is, subsets of D are mapped to subsets of $\mathbb{R}^{n_p}$.

The closed-loop system (9)-(10) resulting from interconnecting $\mathcal{H}_p$ and $\mathcal{H}_c$ is denoted $\mathcal{H}_{cl}$. Its state is given by

$$(\xi, \eta) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_c}$$

and its dynamics by[¶]

$$
\begin{aligned}
\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} &= \begin{bmatrix} f_p(\xi, \kappa_c(\eta, \xi, r)) \\ f_c(\eta, \xi, r) \end{bmatrix} & \left.\begin{array}{l} (\xi, \kappa_c(\eta, \xi, r)) \in C_p \\ \text{and } (\eta, \xi, r) \in C_c \end{array}\right\} , \\[2mm]
\begin{bmatrix} \xi^+ \\ \eta^+ \end{bmatrix} &\in \begin{bmatrix} G_p(\xi, \kappa_c(\eta, \xi, r)) \\ \eta \end{bmatrix} & \left.\begin{array}{l} (\xi, \kappa_c(\eta, \xi, r)) \in D_p \\ \text{and } (\eta, \xi, r) \notin D_c \end{array}\right\} , \\[2mm]
\begin{bmatrix} \xi^+ \\ \eta^+ \end{bmatrix} &\in \begin{bmatrix} \xi \\ G_c(\eta, \xi, r) \end{bmatrix} & \left.\begin{array}{l} (\xi, \kappa_c(\eta, \xi, r)) \notin D_p \\ \text{and } (\eta, \xi, r) \in D_c \end{array}\right\} , \\[2mm]
\begin{bmatrix} \xi^+ \\ \eta^+ \end{bmatrix} &\in \left\{ \begin{bmatrix} G_p(\xi, \kappa_c(\eta, \xi, r)) \\ \eta \end{bmatrix}, \begin{bmatrix} \xi \\ G_c(\eta, \xi, r) \end{bmatrix} \right\} & \left.\begin{array}{l} (\xi, \kappa_c(\eta, \xi, r)) \in D_p \\ \text{and } (\eta, \xi, r) \in D_c \end{array}\right\} .
\end{aligned}
\tag{11}
$$

where, for notational simplicity, we have omitted the argument (t, j) of the time-varying reference r. Solutions to this closed-loop system are denoted $\phi = (\phi_p, \phi_c)$ and are defined as for $\mathcal{H}_p$ earlier.

Using the aforementioned definitions, our goal is to solve the following problem:

Problem ($\star$): *Given a plant $\mathcal{H}_p$ and a complete reference trajectory r, design the data $(C_c, f_c, D_c, G_c, \kappa_c)$ of the controller $\mathcal{H}_c$ so that the error between every plant solution ϕ_p and the reference trajectory r is bounded by a class-$\mathcal{K}_\infty$ function of the difference between their initial values, which corresponds to stability, and asymptotically converging to zero, which corresponds to attractivity.*

The next section formally introduces the notions of stability and attractivity and the proposed approach.

4. A CLASS OF HYBRID CONTROLLERS FOR STATE TRACKING WITH KNOWN REFERENCE TRAJECTORIES

4.1. Main approach

In smooth systems, a well-known approach is to introduce the tracking error $e = \xi - r$ and then analyze the corresponding dynamics. This approach is used for autonomous systems with prespecified time-triggered state jumps in [30]. However, in general, as pointed out in Section 2, the flow and jump sets as well as the flow and jump maps of the error dynamics depend on t and j explicitly, resulting in a nonautonomous hybrid system. To address this issue, we recast Problem ($\star$), which pertains to the stabilization of a time-varying set, as the stabilization of a closed, not necessarily bounded, time-invariant set. Inspired by the idea of treating time in time-varying systems as a state, we achieve this by embedding the given reference trajectory into an extended hybrid system model and defining a set, the *tracking set*, imposing conditions on the state representing tracking of the given reference trajectory. More precisely, given a reference $r : \text{dom } r \to \mathbb{R}^{n_p}$, following (4), we define the set $\mathcal{T}_r$ collecting all of the points (t, j) in the domain of r at which r jumps, that is, every point $(t_j^r, j) \in \text{dom } r$ for which $(t_j^r, j + 1) \in \text{dom } r$. Auxiliary variables $\tau \in \mathbb{R}_{\geqslant 0}$ and $k \in \mathbb{N}$ are incorporated as states to parametrize the given reference trajectory r. In other words, τ evolves continuously according to the flow time parameter t, whereas k evolves discretely according to the

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[¶] For the case when conditions $(\xi, \kappa_c(\eta, \xi, r)) \in D_p$ and $(\eta, \xi, r) \in D_c$ hold simultaneously, the jump map can instead be defined as $\begin{bmatrix} G_p(\xi, \kappa_c(\eta, \xi, r)) \\ G_c(\eta, \xi, r) \end{bmatrix}$. The definition used in (11) leads to an outer semicontinuous set-valued jump map when G_p and G_c are outer semicontinuous. A set-valued map $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *outer semicontinuous* at $x \in \mathbb{R}^n$ if for each sequence $\{x_i\}_{i=1}^\infty$ converging to a point $x \in \mathbb{R}^n$ and each sequence $y_i \in S(x_i)$ converging to a point y, it holds that $y \in S(x)$; see [29, Definition 5.4].

jump time parameter j at jumps of r. In this setting, the set to be stabilized, called *the tracking set*, is given by

$$\mathcal{A} = \{(\xi, \eta, \tau, k) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \times \mathbb{R}_{\geqslant 0} \times \mathbb{N} : \xi = r(\tau, k), \ \eta \in \Phi\}, \tag{12}$$

for some closed set $\Phi \subset \mathbb{R}^{n_c}$ capturing the set of points asymptotically approached by the controller's state. For instance, for the example of Section 2, the resulting tracking set $\mathcal{A}$ with the proposed approach is given by

$$\{(\xi, \eta, \tau, k) : \xi + t_k^r = \tau \in \left[t_k^r, t_{k+1}^r\right], (t_k^r, k) \in (0, 0) \cup \mathcal{T}_r\}, \tag{13}$$

where $\mathcal{T}_r$ is given in (4). The set in (13) is closed and unbounded in the τ and k components.

The next ingredient of the approach is to guarantee, by design of the controller, that the jumps of the plant and of the reference trajectory occur simultaneously. This will be a constraint in the design of the controller, which, although it restricts the type of systems for which the tracking problem can be solved, as Section 6 illustrates, permits to solve a range of relevant hybrid tracking problems.

With a controller satisfying such a property, our approach is to recast the problem under study as a stabilization problem of the set $\mathcal{A}$ for the resulting closed-loop system, which we denote by $\mathcal{H}_{cl}^{\star} = (C, F, G, D)$. Its state is given by

$$x := (\xi, \eta, \tau, k) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \times \mathbb{R}_{\geqslant 0} \times \mathbb{N} =: \mathcal{X}.$$

The flow of $\mathcal{H}_{cl}^{\star}$ is governed by

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\tau} \\ \dot{k} \end{bmatrix} = \begin{bmatrix} f_p(\xi, \kappa_c(\eta, \xi, r(\tau, k))) \\ f_c(\eta, \xi, r(\tau, k)) \\ 1 \\ 0 \end{bmatrix} =: F(\xi, \eta, \tau, k) \tag{14}$$

when flow of $\mathcal{H}_p$, the reference, and $\mathcal{H}_c$ is possible, that is,

$$(\xi, \kappa_c(\eta, \xi, r(\tau, k))) \in C_p, \quad \tau \in [t_k^r, t_{k+1}^r], \quad \text{and} \quad (\eta, \xi, r(\tau, k)) \in C_c. \tag{15}$$

Points (ξ, η, τ, k) satisfying these conditions define the flow set C of $\mathcal{H}_{cl}^{\star}$ as follows:

$$C := \{x \in \mathcal{X} \ : \ (\xi, \kappa_c(\eta, \xi, r(\tau, k))) \in C_p, \tau \in [t_k^r, t_{k+1}^r], (\eta, \xi, r(\tau, k)) \in C_c\}.$$

Jumps of $\mathcal{H}_{cl}^{\star}$ are governed by

$$\begin{bmatrix} \xi^+ \\ \eta^+ \\ \tau^+ \\ k^+ \end{bmatrix} \in \begin{bmatrix} G_p(\xi, \kappa_c(\eta, \xi, r(\tau, k))) \\ \eta \\ \tau \\ k + 1 \end{bmatrix} =: G_1(\xi, \eta, \tau, k) \tag{16}$$

when only the plant and the reference jump, that is,

$$(\xi, \kappa_c(\eta, \xi, r(\tau, k))) \in D_p, \quad (\tau, k) \in \mathcal{T}_r, \quad \text{and} \quad (\eta, \xi, r(\tau, k)) \notin D_c, \tag{17}$$

by

$$\begin{bmatrix} \xi^+ \\ \eta^+ \\ \tau^+ \\ k^+ \end{bmatrix} \in \begin{bmatrix} \xi \\ G_c(\eta, \xi, r(\tau, k)) \\ \tau \\ k \end{bmatrix} =: G_2(\xi, \eta, \tau, k) \tag{18}$$

when only the controller jumps, that is,

$$(\xi, \kappa_c(\eta, \xi, r(\tau, k))) \notin D_p, \quad \tau \in [t_k^r, t_{k+1}^r), \quad \text{and} \quad (\eta, \xi, r(\tau, k)) \in D_c, \tag{19}$$

and by the union of G_1 and G_2 in (16) and (18), respectively, when[||]

$$(\xi, \kappa_c(\eta, \xi, r(\tau, k))) \in D_p, \quad (\tau, k) \in \mathcal{T}_r, \quad \text{and} \quad (\eta, \xi, r(\tau, k)) \in D_c. \tag{20}$$

Points (ξ, η, τ, k) satisfying these conditions define the jump set D of $\mathcal{H}_{cl}^\star$ as follows:

$$D := D_1 \cup D_2, \quad D_1 := \{x \in \mathcal{X} \,:\, (\xi, \kappa_c(\eta, \xi, r(\tau, k))) \in D_p, \, (\tau, k) \in \mathcal{T}_r\},$$
$$D_2 := \{x \in \mathcal{X} \,:\, \tau \in [t_k^r, t_{k+1}^r), \, (\eta, \xi, r(\tau, k)) \in D_c\}.$$

The jump map G of $\mathcal{H}_{cl}^\star$ is given by

$$G(\xi, \eta, \tau, k) := \begin{cases} G_1(\xi, \eta, \tau, k) & (\xi, \eta, \tau, k) \in D_1 \setminus D_2 \\ G_2(\xi, \eta, \tau, k) & (\xi, \eta, \tau, k) \in D_2 \setminus D_1 \\ \{G_1(\xi, \eta, \tau, k), G_2(\xi, \eta, \tau, k)\} & (\xi, \eta, \tau, k) \in D_1 \cap D_2 \end{cases},$$

where G_1 and G_2 are given in (16) and (18), respectively.

The proposed approach reduces the tracking problem to the stabilization of the tracking set $\mathcal{A}$. Then, by exploiting sufficient conditions for asymptotic stability of hybrid systems, a solution to Problem $(\star)$ can be obtained using the stability notion given next. Denoting by $\mathcal{S}_{\mathcal{H}_{cl}^\star}(\phi^\star(0,0))$ the set of maximal solutions $\phi^\star$ to $\mathcal{H}_{cl}^\star$ from $\phi^\star(0,0)$, the set $\mathcal{A}$ is said to be

- *uniformly globally stable* if there exists $\alpha \in \mathcal{K}_\infty$ such that each solution $\phi^\star \in \mathcal{S}_{\mathcal{H}_{cl}^\star}(\phi^\star(0,0))$ satisfies $|\phi^\star(t,j)|_{\mathcal{A}} \leqslant \alpha(|\phi^\star(0,0)|_{\mathcal{A}})$ for all $(t,j) \in \text{dom } \phi^\star$;
- *uniformly globally attractive* if for each $\varepsilon > 0$ and $\lambda > 0$ there exists $N > 0$ such that, for any solution $\phi^\star \in \mathcal{S}_{\mathcal{H}_{cl}^\star}(\phi^\star(0,0))$ with $|\phi^\star(0,0)|_{\mathcal{A}} \leqslant \lambda$, $(t,j) \in \text{dom } \phi^\star$ and $t + j \geqslant N$ imply $|\phi^\star(t,j)|_{\mathcal{A}} \leqslant \varepsilon$; and
- *uniformly globally asymptotically stable* if it is both uniformly globally stable and uniformly globally attractive.

In contrast with the contributions in [13–16], where the controllers only guarantee convergence to zero of the tracking error (i.e., attractivity), the results in the following section provide conditions guaranteeing that the tracking set $\mathcal{A}$ is uniformly globally asymptotically stable.

4.2. Characterization of hybrid controllers

The following result establishes a sufficient condition for stabilization of the tracking set $\mathcal{A}$.

Theorem 4.1 (Sufficient conditions for uniformly globally asymptotically stable)
Given a complete reference trajectory $r : \text{dom } r \to \mathbb{R}^{n_p}$ and associated tracking set $\mathcal{A}$ in (12), if there exists a hybrid controller $\mathcal{H}_c$ guaranteeing that

1. the jumps of r and $\mathcal{H}_p$ occur simultaneously and
2. there exist a function $V : \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \times \mathbb{R}_{\geqslant 0} \times \mathbb{N} \to \mathbb{R}$ that is continuous on $\overline{C \cup D \cup G(D)}$ and locally Lipschitz on a neighborhood of $\overline{C}$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and continuous functions $\rho_1, \rho_2, \rho_3 \in \mathcal{PD}$ such that
   (a) for all $(\xi, \eta, \tau, k) \in \overline{C \cup D \cup G(D)}$,

$$\alpha_1(|(\xi, \eta, \tau, k)|_{\mathcal{A}}) \leqslant V(\xi, \eta, \tau, k) \leqslant \alpha_2(|(\xi, \eta, \tau, k)|_{\mathcal{A}}); \tag{21}$$

[||] Note that with a controller guaranteeing that the jumps of the plant and of the reference trajectory occur simultaneously, we have $(\eta, \xi, r(\tau, k)) \in C_p \Leftrightarrow \tau \in [t_k^r, t_{k+1}^r)$ and $(\xi, \kappa_c(\eta, \xi, r(\tau, k))) \in D_p \Leftrightarrow (\tau, k) \in \mathcal{T}_r$, indicating that writing the conditions involving C_p, D_p, or $\mathcal{T}_r$ is equivalent.

(b) for all $(\xi, \eta, \tau, k) \in C$,

$$V^{\circ}((\xi, \eta, \tau, k), F(\xi, \eta, \tau, k)) \leqslant -\rho_3 \left(|(\xi, \eta, \tau, k)|_{\mathcal{A}} \right); \tag{22}$$

(c) for all $(\xi, \eta, \tau, k) \in D_1$ and all $g \in G_1(\xi, \eta, \tau, k)$,

$$V(g) - V(\xi, \eta, \tau, k) \leqslant -\rho_1 \left(|(\xi, \eta, \tau, k)|_{\mathcal{A}} \right); \tag{23}$$

(d) for all $(\xi, \eta, \tau, k) \in D_2$ and all $g \in G_2(\xi, \eta, \tau, k)$,

$$V(g) - V(\xi, \eta, \tau, k) \leqslant -\rho_2 \left(|(\xi, \eta, \tau, k)|_{\mathcal{A}} \right); \tag{24}$$

then

(1★) there exists $\alpha \in \mathcal{K}_{\infty}$ such that for each $(\phi_p(0,0), \phi_c(0,0)) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_c}$, we have that each maximal solution $\phi = (\phi_p, \phi_c)$ to $\mathcal{H}_{cl}$ in (11) satisfies

$$|(\phi_p(t, j) - r(t, j), \phi_c(t, j))|_{\{0\} \times \Phi} \leqslant \alpha(|(\phi_p(0,0) - r(0,0), \phi_c(0,0))|_{\{0\} \times \Phi}); \tag{25}$$

(2★) for each $\varepsilon > 0$ and each $\lambda > 0$, there exists $N > 0$ such that, for each maximal solution $\phi = (\phi_p, \phi_c)$ to $\mathcal{H}_{cl}$ in (11) with $(\phi_p(0,0), \phi_c(0,0)) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_c}$ such that $|(\phi_p(0,0) - r(0,0), \phi_c(0,0))|_{\{0\} \times \Phi} \leqslant \lambda$, we have that

$$(t, j) \in \operatorname{dom} \phi, \; t + j \geqslant N \quad \Rightarrow \quad |(\phi_p(t, j) - r(t, j), \phi_c(t, j))|_{\{0\} \times \Phi} \leqslant \varepsilon. \tag{26}$$

Proof
Let $\phi = (\phi_p, \phi_c) \in \mathcal{S}_{\mathcal{H}_{cl}}$. For each $(t, j) \in \operatorname{dom} \phi$, define the hybrid arc ϕ_τ on $\operatorname{dom} \phi$ such that $\phi_\tau(t, j) = t$. For each $i \in \mathbb{N}$, denote by $\ell_i \in \mathbb{N}$ the jump times $(t_i^r, \ell_i) \in \operatorname{dom} \phi$ at which the reference jumps, that is, r, have a jump at $(t_0^r, \ell_0), (t_1^r, \ell_1), (t_2^r, \ell_2), \ldots$.** Define the hybrid arc ϕ_k on $\operatorname{dom} \phi$ as

$$\begin{aligned}
\phi_k(t, j) &= 0 & \forall (t, j) \in \operatorname{dom} \phi, t \in [0, t_1^r], j \in \{0, \ldots, \ell_0\}, \\
\phi_k(t, j) &= \phi_k(t_i^r, \ell_{i-1}) + 1 & \forall (t, j) \in \operatorname{dom} \phi, t \in [t_i^r, t_{i+1}^r], j \in \{\ell_{i-1} + 1, \ldots, \ell_i\}, i > 0.
\end{aligned}$$

By using item 1 of the assumptions, by construction,

$$\phi^{\star} := \begin{bmatrix} \phi^{\top}, & \phi_\tau, & \phi_k \end{bmatrix}^{\top} \tag{27}$$

is a maximal solution to $\mathcal{H}_{cl}^{\star}$ from $[\phi(0,0)^{\top}, 0, 0]^{\top}$. Let $t(j)$ denote the smallest time t such that $(t, j) \in \operatorname{dom} \phi^{\star}$ and $j(t)$ denote the smallest index j such that $(t, j) \in \operatorname{dom} \phi^{\star}$. Then, following the arguments in the proof of [26, Theorem 3.18], we have that, for each $(t, j) \in \operatorname{dom} \phi^{\star}$,

$$V(\phi^{\star}(t, j)) - V(\phi^{\star}(0, 0)) = \int_0^t \frac{dV}{d\lambda}(\phi^{\star}(\lambda, j(\lambda))) d\lambda + \sum_{s=1}^j \left[V(\phi^{\star}(t(s), s)) - V(\phi^{\star}(t(s), s-1)) \right].$$

Because along flows, $\phi^{\star}(t, j) \in C$ for each j and every $t \in (t_j, t_{j+1})$ such that $(t, j) \in \operatorname{dom} \phi^{\star}$, using item 2 and the fact that, for almost every $t \in (t_j, t_{j+1})$, $\frac{d}{dt} V(\phi^*(t, j)) \leqslant V^{\circ}(\phi^*(t, j), \dot{\phi}^*(t, j))$ (see [24] and [4, Section IV.B]), we have that for each $t \in (t_j, t_{j+1})$ such that $(t, j) \in \operatorname{dom} \phi^{\star}$,

$$\int_0^t \frac{dV}{d\lambda}(\phi^{\star}(\lambda, j(\lambda))) d\lambda \leqslant -\int_0^t \rho_3(|\phi^{\star}(\lambda, j(\lambda))|_{\mathcal{A}}) d\lambda. \tag{28}$$

Using item 1 and the construction of the jump map G, jumps of the plant/reference and controller are associated with different (t, j)'s in $\operatorname{dom} \phi$. Define $\Gamma_1(\operatorname{dom} \phi^{\star})$ and $\Gamma_2(\operatorname{dom} \phi^{\star})$ as the set

---

**Jumps of the hybrid controller are possible between (t_i^r, ℓ_i) and (t_{i+1}^r, ℓ_{i+1}), and $\ell_{i+1} \geqslant \ell_i + 1$ because jumps of the plant occur when r jumps, as given in item 1.

of such (t, j)'s, respectively (in particular, $\Gamma_1(\text{dom } \phi^\star)$ is the collection of points (t_i^r, ℓ_i) defined earlier). By using item 2, it follows that for each $(t', j') \in \Gamma_1(\text{dom } \phi^\star)$, because only the plant and reference have a jump, we have

$$V(\phi^\star(t', j' + 1)) - V(\phi^\star(t', j')) \leqslant -\rho_1\left(|\phi^\star(t', j')|_{\mathcal{A}}\right). \tag{29}$$

By using item 2, it follows that for every $(t', j') \in \Gamma_2(\text{dom } \phi^\star)$, only the controller jumps, and we have

$$V(\phi^\star(t', j' + 1)) - V(\phi^\star(t', j')) \leqslant -\rho_2\left(|\phi^\star(t', j')|_{\mathcal{A}}\right). \tag{30}$$

Combining bounds (28)–(30), we obtain, for each $(t, j) \in \text{dom } \phi^\star$,

$$
\begin{aligned}
V(\phi^\star(t, j)) - V(\phi^\star(0, 0)) \leqslant & -\int_0^t \rho_3(|\phi^\star(\lambda, j(\lambda))|_{\mathcal{A}})\mathrm{d}\lambda \\
& - \sum_{(t', j') \in \Gamma_1(\text{dom } \phi^\star), t'+j' \leqslant t+j} \rho_1\left(|\phi^\star(t', j')|_{\mathcal{A}}\right) \\
& - \sum_{(t', j') \in \Gamma_2(\text{dom } \phi^\star), t'+j' \leqslant t+j} \rho_2\left(|\phi^\star(t', j')|_{\mathcal{A}}\right).
\end{aligned}
\tag{31}
$$

Because of ρ_1, ρ_2, ρ_3 taking on nonnegative values, we already have

$$V(\phi^\star(t, j)) \leqslant V(\phi^\star(0, 0)) \qquad \forall (t, j) \in \text{dom } \phi^\star.$$

By the definition of hybrid arc in Section 3 and completeness of r, $\mathcal{A}$ is a closed subset of $\mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \times \mathbb{R}_{\geqslant 0} \times \mathbb{N}$. Using the lower bound in (21), we have

$$|(\phi(t, j), \phi_\tau(t, j), \phi_k(t, j))|_{\mathcal{A}} \leqslant \alpha_1^{-1} \circ V(\phi^\star(0, 0)) = \alpha_1^{-1} \circ V(\phi(0, 0), 0, 0) \qquad \forall (t, j) \in \text{dom } \phi^\star,$$

and using the upper bound, we obtain

$$
\begin{aligned}
|(\phi(t, j), \phi_\tau(t, j), \phi_k(t, j))|_{\mathcal{A}} &\leqslant \alpha_1^{-1} \circ V(\phi(0, 0), 0, 0) \\
&\leqslant \alpha_1^{-1} \circ \alpha_2(|(\phi_p(0, 0) - r(0, 0), \phi_c(0, 0))|_{\{0\} \times \Phi})
\end{aligned}
$$

for all $(t, j) \in \text{dom } \phi^\star$. Then, $\mathcal{A}$ is uniformly globally stable for $\mathcal{H}_{cl}^\star$. By the definition of $\mathcal{A}$, we have that (25) holds with $\alpha := \alpha_1^{-1} \circ \alpha_2$.

Now, we show uniform global attractivity. Given ε and λ satisfying $0 < \varepsilon < \lambda$, let ε' be such that $0 < \varepsilon' < \varepsilon$. Define

$$\Omega(\varepsilon', \lambda) := \{z \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \times \mathbb{R}_{\geqslant 0} \times \mathbb{N} : \varepsilon' \leqslant |z|_{\mathcal{A}} \leqslant \lambda\},$$

and let $\lambda' > 0$ be such that $\{z : V(x) \leqslant \lambda'\} \subset \Omega(0, \lambda)$. For each solution $\phi = (\phi_p, \phi_c) \in \mathcal{S}_{\mathcal{H}_{cl}}$ with $(\phi_p(0, 0), \phi_c(0, 0), 0, 0) \in \{z : V(z) \leqslant \lambda'\}$, following the construction in (27), associate a solution $\phi^\star$ to $\mathcal{H}_{cl}^\star$. It follows that each $\phi^\star$ satisfies (31), and because of the right-hand side of this expression being nonpositive, each $\phi^\star$ remains in $\Omega(0, \lambda)$. Using (21), we obtain from (31)

$$\alpha_1(|\phi^\star(t, j)|_{\mathcal{A}}) \leqslant V(\phi^\star(t, j)) \leqslant -\rho(\varepsilon')(t + j_1(t, j) + j_2(t, j)) + \alpha_2(\lambda) \tag{32}$$
$$\leqslant -\rho(\varepsilon')(t + j) + \alpha_2(\lambda) \tag{33}$$

for each $(t, j) \in \text{dom } \phi^\star$ for which $\phi^\star(t, j) \in \Omega(\varepsilon', \lambda)$, where $\rho(s) := \min\{\rho_1(s), \rho_2(s), \rho_3(s)\}$ for each $s \geqslant 0$, and for each $i = 1, 2$, $j_i(t, j)$ denotes the largest j' such that $(t', j') \in \Gamma_i(\text{dom } \phi^\star)$ and $t' + j' \leqslant t + j$. To arrive to (33), we have used the fact that, by construction, $j_1(t, j) + j_2(t, j) = j$. Then, it follows that

$$t + j \geqslant N := \frac{\alpha_2(\lambda) - \alpha_1(\varepsilon)}{\rho(\varepsilon')} \qquad \Rightarrow \qquad |\phi^\star(t, j)|_{\mathcal{A}} \leqslant \varepsilon.$$

Consequently, $\mathcal{A}$ is uniformly globally attractive for $\mathcal{H}_{cl}^\star$. By the definition of $\mathcal{A}$, we have that (26) holds.   □

Remark 4.2
According to Theorem 4.1, the hybrid controller $\mathcal{H}_c$ has to be synthesized so that it enforces (22)-(24). In particular, condition (22) depends on f_c, C_c and κ_c; (23) depends on κ_c; and (24) depends on G_c and D_c, which are all to be chosen in the design. These can be exploited in deriving constructive control design techniques for specific classes of hybrid systems. The next section introduces a synthesis procedure for a class of hybrid systems with linear maps and input-triggered jumps. Several examples in Section 6 illustrate the feasibility of the design of controllers satisfying the conditions of the theorem.

Remark 4.3
Theorem 4.1 implies that complete solutions to the closed-loop system are such that $|(\xi, \eta, \tau, k)(t, j)|_{\mathcal{A}} \to 0$ as $t + j \to \infty$, that is,

$$|\xi(t, j) - r(\tau(t, j), k(t, j))| \to 0 \qquad \text{as } t + j \to \infty.$$

This includes all possible solutions with unconstrained initial conditions of τ and k, in particular, it holds for all solutions with $\tau(0, 0) = k(0, 0) = 0$, for which $r(\tau(t, j), k(t, j)) = r(t, j)$ and, consequently,

$$|\xi(t, j) - r(t, j)| \to 0 \qquad \text{as } t + j \to \infty.$$

Note that complete solutions to $\mathcal{H}_{cl}^{\star}$ have the property that $\tau(t, j) + k(t, j)$ is unbounded as $t + j \to \infty$. Furthermore, it implies that $\xi(t, j) = r(t, j)$ on the domain of definition of solutions starting from $\xi(0, 0) = r(0, 0), \tau(0, 0) = k(0, 0) = 0$, when solutions from such points exist.

Conditions 2(b)–(d) in Theorem 4.1 can be relaxed under further conditions on the solutions to $\mathcal{H}_{cl}$. In particular, the next result following [26, Proposition 3.24] relaxing the conditions at jumps will be useful in designing a tracking controller for a motion control system in Example 6.2. It establishes that conditions $(1\star)$ and $(2\star)$ in Theorem 4.1 hold if the Lyapunov function is nonincreasing during jumps, strictly decreasing during flow, and the duration of flow is sufficiently large for every solution.

Corollary 4.4 (Special case of sufficient conditions for uniformly globally asymptotically stable)
Suppose that the conditions in Theorem 4.1 hold with $\rho_1 \equiv 0$ and $\rho_2 \equiv 0$. If for each $\lambda > 0$, there exists $\gamma \in \mathcal{K}_\infty$ such that for every maximal solution ϕ to $\mathcal{H}_{cl}$ in (11) with $|(\phi_p(0, 0) - r(0, 0), \phi_c(0, 0))|_{\{0\} \times \Phi} \leqslant \lambda$,

$$t + j \geqslant N, (t, j) \in \operatorname{dom} \phi \qquad \Rightarrow \qquad t \geqslant \gamma(N), \tag{34}$$

then $(1\star)$ and $(2\star)$ in Theorem 4.1 hold.

Proof
The claim follows from (32) with $\rho = \rho_3$ and (34). In fact, continuing the proof of Theorem 4.1 from (32), we obtain

$$\alpha_1(|\phi^{\star}(t, j)|_{\mathcal{A}}) \leqslant V(\phi^{\star}(t, j)) \leqslant -\rho_3(\varepsilon')(t + j_1(t, j) + j_2(t, j)) + \alpha_2(\lambda) \tag{35}$$
$$\leqslant -\rho_3(\varepsilon')t + \alpha_2(\lambda). \tag{36}$$

Then, $|\phi^{\star}(t, j)_{\mathcal{A}} \leqslant \varepsilon$ holds when $t + j \geqslant N := \gamma^{-1}\left(\frac{\alpha_2(\lambda) - \alpha_1(\varepsilon)}{\rho(\varepsilon')}\right)$. $\qquad\square$

5. SYNTHESIS RESULTS FOR A CLASS OF HYBRID SYSTEMS WITH LINEAR MAPPINGS

In Section 4, we have presented a characterization of tracking controllers solving Problem $(\star)$ for a general class of hybrid systems. This characterization, formalized in Theorem 4.1, requires the

tracking controller to ensure that the plant and the reference jump simultaneously as well as that the tracking set is asymptotically stabilized. This section presents controller synthesis results for a particular class of hybrid systems with linear flow and jump maps in terms of constructive conditions in the form of LMIs. As in related works (see, e.g, [31, 32]), impulsive control inputs are permitted when controlling the plants.

Let us consider a class of plants given by the hybrid systems of the form

$$\mathcal{H}_p^\ell \begin{cases} \dot{\xi} = f_p(\xi, u) := A_p \xi + B_p u_1 & (\xi, u) \in C_p := \{(\xi, u) : u_2 = 0\} \\ \xi^+ = G_p(\xi, u) := E_p \xi + F_p u_2 & (\xi, u) \in D_p := \{(\xi, u) : u_2 \neq 0\} , \\ y = h(\xi) := \xi \end{cases} \quad (37)$$

where $\xi \in \mathbb{R}^{n_p}$ and $u_1, u_2 \in \mathbb{R}^{m_p}$. In (37), we employ a decomposition of the control input $u = \begin{bmatrix} u_1^\top & u_2^\top \end{bmatrix}^\top \in \mathbb{R}^{2m_p}$ into an impulsive control component u_2 and a nonimpulsive control component u_1. Note that, compared with (9), we now consider plants with single-valued jump map G_p. In the next section, we study an example of a mechanical motion system with impulsive controls that can be modeled as in (37).

The following assumption imposes the existence of feedforward inputs $u^r = \begin{bmatrix} u_1^{r\top} & u_2^{r\top} \end{bmatrix}^\top$ inducing the reference r to be tracked.

Assumption 5.1
The reference trajectory r has a sequence of jump times $\mathcal{T}_r$, and there exist inputs u_1^r and u_2^r such that

$$\begin{aligned} \dot{r} &= A_p r + B_p u_1^r & (t, j) \notin \mathcal{T}_r, \\ r^+ &= E_p r + F_p u_2^r & (t, j) \in \mathcal{T}_r, \end{aligned} \quad (38)$$

and $u_2^r(t, j) \neq 0$ if and only if $(t, j) \in \mathcal{T}_r$.

We propose the following family of static controllers to solve the tracking problem:

$$\begin{aligned} u_1 &= u_1^r + K_1(\xi - r), \\ u_2 &= \begin{cases} 0 & u_2^r = 0 \\ u_2^r + K_2(\xi - r) & u_2^r \neq 0 \text{ and } u_2^r + K_2(\xi - r) \neq 0 , \\ u_2^r + (1 + \varepsilon)K_2(\xi - r) & u_2^r \neq 0 \text{ and } u_2^r + K_2(\xi - r) = 0 \end{cases} \end{aligned} \quad (39)$$

where $\varepsilon > 0$ is a sufficiently small parameter and K_1, K_2 are controller gains to be designed. During jumps, the feedback law u_2 updates the plant state by using a linear function of the state and reference. Moreover, the expression for u_2 (especially, the third line of its definition in (39)) enforces that

$$u_2 = 0 \text{ if and only if } u_2^r = 0, \quad (40)$$

which, when combined with the conditions in Assumption 5.1, implies that jumps of the plant always coincide with jumps of the reference trajectory. Finally, control law (39) can be decomposed into feedforward terms (u_1^r and u_2^r) and feedback terms depending on the tracking error ($\xi - r$).

Incorporating the auxiliary variables (τ, k) as in Section 4.1, because of the static linear controller (39), the closed-loop system $\mathcal{H}_{cl}^\star$ results in

$$\mathcal{H}_{cl}^{\star,\ell} \begin{cases} \begin{rcases} \dot{\xi} = A_p \xi + B_p \left(u_1^r(\tau, k) + K_1(\xi - r(\tau, k)) \right) \\ \dot{\tau} = 1 \\ \dot{k} = 0 \end{rcases} \tau \in \left[t_k^r, t_{k+1}^r \right], k \in \mathbb{N} \\ \begin{rcases} \xi^+ = G_{p,cl}(\xi, \tau, k) \\ \tau^+ = \tau \\ k^+ = k + 1 \end{rcases} (\tau, k) \in \mathcal{T}_r \end{cases} , \quad (41)$$

where

$$G_{p,cl}(\xi,\tau,k) = \begin{cases} E_p\xi+F_p\big(u_2^r(\tau,k)+K_2(\xi-r(\tau,k))\big) & u_2^r(\tau,k)+K_2(\xi-r(\tau,k)) \neq 0 \\ E_p\xi+F_p\big(u_2^r(\tau,k)+(1+\varepsilon)K_2(\xi-r(\tau,k))\big) & u_2^r(\tau,k)+K_2(\xi-r(\tau,k)) = 0 \end{cases}.$$

We provide conditions to design the feedback gains K_1, K_2 and the parameter ε in (39) such that the tracking set $\mathcal{A}$ in (12) (which, because of the static controller considered here, is given by $\mathcal{A} = \{(\xi,\tau,k) \in \mathbb{R}^{n_p} \times \mathbb{R}_{\geqslant 0} \times \mathbb{N} : \xi - r(\tau,k) = 0\}$) is uniformly globally asymptotically stable for closed-loop system (41). Our first synthesis result is as follows.

Theorem 5.2
Consider a hybrid plant given by (37) and a complete hybrid reference trajectory r. If Assumption 5.1 holds and the following LMIs are feasible for real matrices Y_1, Y_2 and a real symmetric matrix X

$$A_p X + B_p Y_1 + X A_p^\top + Y_1^\top B_p^\top \prec 0, \tag{42}$$

$$\begin{pmatrix} X & XE_p^\top + Y_2^\top F_p^\top \\ E_p X + F_p Y_2 & X \end{pmatrix} \succ 0, \tag{43}$$

then tracking controller (39) with gains

$$K_1 = Y_1 X^{-1}, \quad K_2 = Y_2 X^{-1}$$

and $\varepsilon > 0$ such that[††]

$$\begin{pmatrix} X & XE_p^\top + Y_2^\top F_p^\top \\ E_p X + F_p Y_2 & X \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & Y_2^\top F_p^\top \\ F_p Y_2 & 0 \end{pmatrix} \succ 0, \tag{44}$$

guarantees that the tracking set $\mathcal{A}$ is uniformly globally asymptotically stable for closed-loop system (41).

Proof
Note that the dynamics of the tracking error $e := \xi - r(\tau,k)$ are given by

$$\begin{aligned} \dot{e} &= (A_p + B_p K_1)e =: f_e(e), & \tau \in [t_k^r, t_{k+1}^r], k \in \mathbb{N}, \\ e^+ &= (E_p + F_p K_2)e =: g_{e,1}(e), & (\tau,k) \in \mathcal{T}_r \text{ and } u_2^r(\tau,k) + K_2 e \neq 0, \\ e^+ &= (E_p + F_p(1+\varepsilon)K_2)e =: g_{e,2}(e), & (\tau,k) \in \mathcal{T}_r \text{ and } u_2^r(\tau,k) + K_2 e = 0. \end{aligned} \tag{45}$$

Consider a candidate Lyapunov function $V(e) = e^\top P e$ with symmetric matrix P for system (45). Its evaluation along solutions of (45) gives

$$\begin{aligned} \langle \nabla V(e), f_e(e) \rangle &= e^\top \left(P(A_p + B_p K_1) + (A_p + B_p K_1)^\top P \right) e & (\tau,k) \notin \mathcal{T}_r, \\ V(g_{e,1}(e)) - V(e) &= e^\top \left((E_p + F_p K_2)^\top P(E_p + F_p K_2) - P \right) e & (\tau,k) \in \mathcal{T}_r \wedge u_2^r(\tau,k) + K_2 e \neq 0, \\ V(g_{e,2}(e)) - V(e) &= e^\top \left((E_p + F_p(1+\varepsilon)K_2)^\top P(E_p + F_p(1+\varepsilon)K_2) - P \right) e \\ & \qquad\qquad\qquad (\tau,k) \in \mathcal{T}_r \wedge u_2^r(\tau,k) + K_2 e = 0. \end{aligned} \tag{46}$$

Theorem 4.1 implies that $\mathcal{A}$ is globally asymptotically stable for (41) if the following matrix inequalities hold:

$$P \succ 0, \tag{47}$$

[††] Condition (43) implies that condition (44) holds for sufficiently small parameter ε.

$$P(A_p + B_p K_1) + (A_p + B_p K_1)^\top P \prec 0, \tag{48}$$

$$(E_p + F_p K_2)^\top P(E_p + F_p K_2) - P \prec 0, \tag{49}$$

$$(E_p + F_p(1 + \varepsilon)K_2)^\top P(E_p + F_p(1 + \varepsilon)K_2) - P \prec 0. \tag{50}$$

In fact, symmetry of the matrix P implies that (21) holds with $V(e) = e^\top P e$, $\alpha_1(s) = \lambda_{\min}(P)s^2$ and $\alpha_2(s) = \lambda_{\max}(P)s^2$, where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are strictly positive because of (47). Moreover, (48) implies the existence of a positive definite matrix $Q_3 \succ 0$ such that

$$\langle \nabla V(e), f_e(e) \rangle \leqslant -e^\top Q_3 e \leqslant -\lambda_{\min}(Q_3)|e|^2$$

holds. Hence, (22) holds with $\rho_3(s) = \lambda_{\min}(Q_3)s^2$. Additionally, (49) and (50) imply the existence of a positive definite matrix $Q_1, > 0$ such that

$$V(g_{e,i}(e)) - V(e) \leqslant -e^\top Q_1 e \leqslant -\lambda_{\min}(Q_1)|e|^2, \text{ for } i = 1, 2,$$

from which (23) follows with $\rho_1(s) = \lambda_{\min}(Q_1)s^2$. We can select $\rho_1 = \rho_2$ because requirement 2 of Theorem 4.1 holds vacuously: static controller (39) does not contain jumps of internal states.

Using $P \succ 0$ and applying a Schur complement to (49) give

$$\begin{pmatrix} P & E_p^\top + K_2^\top F_p^\top \\ E_p + F_p K_2 & P^{-1} \end{pmatrix} \succ 0. \tag{51}$$

By premultiplying and postmultiplying this inequality by $\begin{bmatrix} P^{-1} & 0 \\ 0 & I \end{bmatrix}$, we obtain the equivalent expression

$$\begin{pmatrix} P^{-1} & P^{-1}E_p^\top + P^{-1}K_2^\top F_p^\top \\ E_p P^{-1} + F_p K_2 P^{-1} & P^{-1} \end{pmatrix} \succ 0. \tag{52}$$

By using the change of variables $X = P^{-1}$ and $Y_2 = K_2 P^{-1}$, the aforementioned inequality can be rewritten as (43) (where (43) implies that X and P are indeed positive definite). Observe that (49) directly implies (50) (as well as its equivalent form in (44)) for sufficiently small $\varepsilon > 0$ because the eigenvalues of the parameterized matrix $(E_p + F_p(1 + \tilde{\varepsilon})K_2)^\top P(E_p + (1 + \tilde{\varepsilon})F_p K_2) - P$ with $\tilde{\varepsilon} \geqslant 0$ small are continuous in $\tilde{\varepsilon}$ (see, e.g., [33, Theorem 5.1, p. 107]).

Premultiplying and postmultiplying (48) by P^{-1} give equivalently

$$P^{-1}A_p^\top + P^{-1}K_1^\top B_p^\top + A_p P^{-1} + B_p K_1 P^{-1} \prec 0. \tag{53}$$

Applying the change of variables $X = P^{-1}$ and $Y_1 = K_1 P^{-1}$ gives LMI (42). This completes the proof.   □

The conditions in Theorem 5.2 require the existence of a (quadratic) Lyapunov function that decays both along flows and jumps of the closed-loop system. Because this requirement may be difficult to meet in some cases, we now formulate a result that alleviates this requirement to requirements where only decay of the Lyapunov function along either flow or jumps is required together with additional requirements on the set $\mathcal{T}_r$ of reference jump times. These requirements are formulated by adapting the formulation of the jump time requirements of [34] to our case, where multiple jumps can occur at the same time instant. This leads to the following definitions of *minimum average interjump time* and *maximal average interjump time*. Related contributions in the context of general stability (not necessarily for tracking) allowing for increase of Lyapunov functions include the work in [35] for dynamical systems with solutions exhibiting jumps and in [36] for the stabilization of rigid bodies by means of hybrid control.

Definition 5.3 • The sequence of jump times $\{(t_j^r, j^r)\}$ in $\mathcal{T}_r$ is said to have *minimal average interjump time* $\tau > 0$, if there exists an $N_0 > 0$ such that for all $(t, j) \in \text{dom } r$ and all $(T, J) \in \text{dom } r$ where $T + J \geqslant t + j$,

$$J - j \leqslant N_0 + \frac{T - t}{\tau}. \tag{54}$$

All jump time sequences $\{(t_j^r, j^r)\}$ such that (54) is satisfied are denoted by $\mathcal{S}_{\text{avg}}[\tau, N_0]$.

• The sequence of jump times $\{(t_j^r, j^r)\}$ in $\mathcal{T}_r$ is said to have *maximal average interjump time* $\tau > 0$, if there exists an $N_0 > 0$ such that for all $(t, j) \in \text{dom } r$ and all $(T, J) \in \text{dom } r$ where $T + J \geqslant t + j$,

$$J - j \geqslant \frac{T - t}{\tau} - N_0. \tag{55}$$

All jump time sequences $\{(t_j^r, j^r)\}$ such that (55) is satisfied are denoted by $\mathcal{S}_{\text{r-avg}}[\tau]$.

As observed in [34], $\mathcal{S}_{\text{avg}}[\tau, 1]$ corresponds to a minimal interjump time requirement with dwell-time τ. For $N_0 > 1$, $\{(t_j^r, j^r)\} \in \mathcal{S}_{\text{avg}}[\tau, N_0]$ implies that the sequence of jump times satisfies an average interjump time requirement. Given $N_0 > 0$, condition (55) is a *reverse average interjump time* condition that demands jumps to occur in average at least every τ time units.

Now, we can state the following result.

Theorem 5.4
Consider a hybrid plant given by (37) and a complete hybrid reference trajectory r. Suppose Assumption 5.1 holds and that either of the following two conditions holds:

(A.1) The sequence of reference jump times $\{(t_j^r, j^r)\}$ is in $\mathcal{S}_{\text{avg}}[\tau, N_0]$ with $\tau \geqslant -\frac{d}{c}$, $c > 0$, $d < 0$ and $N_0 > 0$.

(A.2) The sequence of reference jump times $\{(t_j^r, j^r)\}$ is in $\mathcal{S}_{\text{r-avg}}[\tau]$ with $\tau \leqslant -\frac{d}{c}$, $c < 0$, $d > 0$, $N_0 > 0$ and dom r is unbounded in t-direction.

If, additionally, the following LMIs are feasible for real matrices Y_1, Y_2 and a real symmetric matrix X:

$$A_p X + B_p Y_1 + X A_p^\top + Y_1^\top B_p^\top + c X \prec 0, \tag{56}$$

$$\begin{pmatrix} \exp(-d)X & X E_p^\top + Y_2^\top F_p^\top \\ E_p X + F_p Y_2 & X \end{pmatrix} \succ 0, \tag{57}$$

then tracking controller (39) with gains $K_1 = Y_1 X^{-1}$, $K_2 = Y_2 X^{-1}$, and $\epsilon > 0$ such that

$$\begin{pmatrix} \exp(-d)X & X E_p^\top + Y_2^\top F_p^\top \\ E_p X + F_p Y_2 & X \end{pmatrix} + \epsilon \begin{pmatrix} 0 & Y_2^\top F_p^\top \\ F_p Y_2 & 0 \end{pmatrix} \succ 0, \tag{58}$$

guarantees that the set $\mathcal{A}$ is a uniformly globally asymptotically stable set for closed-loop system (41).

Proof
Similar to the proof of Theorem 5.2, we prove uniform global asymptotic stability of $\mathcal{A}$ for dynamics (41) by proving that the set $e = 0$ is uniformly globally asymptotically stable for (45). Along the lines of Theorem 5.2, it can be shown that the satisfaction of LMIs (56) and (58) guarantees that the evolution of the candidate Lyapunov function $V(e) = e^\top P e$, with $P = P^\top \succ 0$, along the solutions of (45) satisfies

$$\langle \nabla V(e), f_e(e) \rangle \leqslant -(c + \delta)V \qquad (t, j) \notin \mathcal{T}_r, \tag{59}$$

$$V(g_{e,i}(e)) \leqslant \exp(-d)V \qquad (t, j) \in \mathcal{T}_r, \text{ for } i = 1, 2 \tag{60}$$

for some $\delta > 0$.

Let us now show that (59) and (60) imply that $\mathcal{A}$ is a uniformly globally asymptotically stable set of system (41) if either of the two conditions on the sequence of reference jump times $\{(t_j^r, j^r)\}$ posed in the theorem, that is, condition A.1 or A.2, holds. Here, we apply analogous reasoning as presented in [34], further taking into account that jumps can directly follow each other.

Let us first consider the case in which condition A.1 holds, and let $T \geqslant 0$ be given. Because this condition requires a minimum average interjump time, it follows that the time domain dom r is unbounded in t-direction, that is, for all $T \geqslant 0$, there exists $J \in \mathbb{N}$ such that $(T, J) \in$ dom r. By construction of controller (39) and Assumption 5.1, the time domain of (41) is equal to dom r, such that $e(T, J)$ can be evaluated for all $T \geqslant 0$ and J such that $(T, J) \in$ dom r. From (59) and (60), with $c > 0$ and $d < 0$, it follows that

$$V(e(T, J)) \leqslant \exp(-cT - \delta T) \exp(-dJ) V(e(0, 0)), \text{ for } (T, J) \in \text{dom } r. \qquad (61)$$

The fact that the sequence of reference jump times $\{(t_j^r, j^r)\}$ is in $\mathcal{S}_{\text{avg}}[\tau, N_0]$ implies that $J \leqslant N_0 + \frac{T}{\tau}$ and, hence,

$$V(e(T, J)) \leqslant \exp(-dN_0) \exp\left(\left(-c - \frac{d}{\tau} - \delta\right) T\right) V(e(0, 0)), \text{ for } (T, J) \in \text{dom } r, \qquad (62)$$

where we also used that $d < 0$. Next, we use that $\tau \geqslant -\frac{d}{c}$, with $c > 0$, implies that $-c - \frac{d}{\tau} - \delta \leqslant -\delta < 0$, which together with (62) and the fact $V(e) = e^{\top} P e$ with $P = P^{\top} \succ 0$, indeed implies that the set $e = 0$ is uniformly globally asymptotically stable under dynamics (45). Hence, $\mathcal{A}$ is uniformly globally asymptotically stable for the closed-loop system (41).

Next, let us consider the case in which condition A.2 holds, and let $T \geqslant 0$ be given. Because dom r is unbounded in t-direction, for all $T \geqslant 0$, we can choose a J such that $(T, J) \in$ dom r, and one can evaluate $e(T, J)$ for $T \to \infty$ and appropriate J.

From (60) and (59) with $c < 0$ and $d > 0$, it follows that

$$V(e(T, J)) \leqslant \exp\left(-cT - \delta T\right) \exp\left(-dJ\right) V(e(0, 0)), \text{ for } (T, J) \in \text{dom } r. \qquad (63)$$

The fact that the sequence of reference jump times $\{(t_j^r, j^r)\}$ is in $\mathcal{S}_{\text{r-avg}}[\tau, N_0]$ implies that $J \geqslant \frac{T}{\tau} - N_0$ and, hence,

$$V(e(T, J)) \leqslant \exp\left(\left(-c - \frac{d}{\tau} - \delta\right) T\right) V(e(0, 0)), \text{ for } (T, J) \in \text{dom } r, \qquad (64)$$

where we also used that $d > 0$. Next, we use that $\tau \leqslant -\frac{d}{c}$, with $c < 0$ and $d > 0$, implies that $-c - \frac{d}{\tau} - \delta \leqslant -\delta < 0$, which together with (64) and the fact $V(e) = e^{\top} P e$, with $P = P^{\top} \succ 0$, indeed implies that the set $e = 0$ is uniformly globally asymptotically stable for system (45). Hence, $\mathcal{A}$ is uniformly globally asymptotically stable for closed-loop system (41). □

It is worth pointing out that, similarly to Theorem 5.2, condition (57) in Theorem 5.4 implies that condition (58) holds for sufficiently small parameter ε. In this way, once condition (57) is satisfied, the design of the parameter ε reduces to picking a sufficiently small value such that (58) holds.

6. EXAMPLES

In this section, first, we illustrate our results in Section 4.2 and show that in the presented examples, a given controller solves the tracking problem. Subsequently, in three examples, the synthesis results presented in Section 5 are used to design controllers solving the tracking problem.

Example 6.1 (Tracking a periodic square wave signal)
Consider the scalar hybrid plant $\mathcal{H}_p$

$$\dot{\xi} = -a\xi + u_1 \qquad \xi\,u_1 \geqslant 0, |\xi| > 0, \tag{65}$$

$$\xi^+ = b + u_2 \qquad \xi\,u_1 \leqslant 0, |\xi| > 0, \tag{66}$$

where[‡‡] $a, b > 0$, and consider the problem of tracking the square wave signal

$$r(t, j) = (-1)^{j+1}$$

defined for each (t, j) such that $t \in [t_j^r, t_{j+1}^r]$, $t_j^r = j$, $j \in \mathbb{N}$. It follows that $\mathcal{T}_r := \{(1, 0), (2, 1), (3, 2), \ldots\}$. Considering static state-feedback controllers and following the approach proposed in Section 4, the goal is to solve Tracking Control Problem ($\star$) with $\mathcal{A}$ given by

$$\mathcal{A} = \{(\xi, \tau, k) \,:\, \xi = r(\tau, k)\} = \{(\xi, \tau, k) \,:\, \xi = (-1)^{k+1}, \tau \in [t_k^r, t_{k+1}^r], (t_k^r, k) \in (0, 0) \cup \mathcal{T}_r\}.$$

For this purpose, we consider the controller

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \kappa_c(\xi, r(\tau, k)) = \begin{bmatrix} a\,r(\tau, k) \\ -b - r(\tau, k) + \lambda(\xi - r(\tau, k)) \end{bmatrix},$$

with $\lambda \in [0, 1)$. It follows that, for every initial condition $\xi(0, 0) < 0$, every jump of r triggers a jump of the plant. In fact, if $\xi(0, 0) < 0$, because $u_1 = a\,r(\tau, k)$, we have that $a\xi(0, 0)r(0, 0) > 0$ and solutions initially flow. Without loss of generality, we only consider trajectories from initial conditions $\xi(0, 0) < 0$ because trajectories of $\xi(0, 0) > 0$ would directly experience a jump at $(t, j) = (0, 0)$, because $r(0, 0) = -1$.

Flows of ξ will not trigger a jump because the sign of ξ remains constant. Jumps of the closed-loop system occur only when r changes sign, which occurs when $(t, j) \in \mathcal{T}_r$.

Then, the resulting closed-loop system is given by

$$\left.\begin{array}{rcl} \dot{\xi} & = & -a(\xi + r(\tau, k)) \\ \dot{\tau} & = & 1 \\ \dot{k} & = & 0 \end{array}\right\} \begin{array}{c} a\,\xi\,r(\tau, k) \geqslant 0, |\xi| > 0 \\ \tau \in [t_k^r, t_{k+1}^r] \end{array}, \tag{67}$$

$$\left.\begin{array}{rcl} \xi^+ & = & -r(\tau, k) + \lambda(\xi - r(\tau, k)) \\ \tau^+ & = & \tau \\ k^+ & = & k + 1 \end{array}\right\} \begin{array}{c} a\,\xi\,r(\tau, k) \leqslant 0, |\xi| > 0 \\ (\tau, k) \in \mathcal{T}_r \end{array}, \tag{68}$$

which can be written as $\mathcal{H}$ with data (C, f, D, G) and captures all of the solutions to the original system with initial conditions $\xi(0, 0) < 0$, $\tau(0, 0) = k(0, 0) = 0$. To establish asymptotic stability of $\mathcal{A}$ for (67) and (68), consider the Lyapunov function

$$V(\xi, \tau, k) = \frac{1}{2}(\xi - r(\tau, k))^2,$$

for which condition (21) holds trivially because for each $(\xi, \tau, k) \in C \cup D \cup G(D)$, $|(\xi, \tau, k)|_{\mathcal{A}} = |\xi - r(\tau, k)|$.[§§] For each (ξ, τ, k) satisfying $a\,\xi\,r(\tau, k) \geqslant 0, |\xi| > 0, \tau \in [t_k^r, t_{k+1}^r]$, we have

$$\langle \nabla V(\xi, \tau, k), f(\xi, \tau, k) \rangle = -a(\xi - r(\tau, k))^2 - (\xi - r(\tau, k))\frac{dr}{d\tau}(\tau, k) = -2aV(\xi, \tau, k)$$

[‡‡]The condition $|\xi| > 0$ in the flow and jump sets of $\mathcal{H}_p$ removes solutions that only jump at the origin. In fact, otherwise, for the initial condition $\xi(0, 0) = 0$, there exists an input u_2 such that there is a solution ϕ_p satisfying $\phi_p(0, j) = 0$ for all $j \in \mathbb{N}$.

[§§] This follows by noting that, for the particular reference trajectory, the minimizer $x' = (\xi', \tau', k')$ of $\min_{x' \in \mathcal{A}} |x - x'|$ for each $x \in C \cup D \cup G(D)$ must satisfy $k' = k$. Then, the minimum distance between (ξ, τ, k) and $\mathcal{A}$ is the distance between a point and a line, which in this case is given by $|\xi - r(\tau, k)|$.

because $\frac{\partial r}{\partial \tau} = 0$ for all τ, k. For each (ξ, τ, k) satisfying $a\,\xi\,r(\tau, k) \leqslant 0, |\xi| > 0, (\tau, k) \in \mathcal{T}_r$, we have

$$V(G(\xi, \tau, k)) - V(\xi, \tau, k) = -(1 - \lambda^2)V(\xi, \tau, k).$$

Because $a > 0$ and $\lambda \in [0, 1)$, Theorem 4.1 implies uniform global asymptotic stability of $\mathcal{A}$ for closed-loop systems (67) and (68) (actually, the aforementioned Lyapunov function V establishes exponential stability). Figure 3(a) depicts a closed-loop system trajectory converging to the reference asymptotically, both along flows and jumps. Figure 3(b) depicts the Lyapunov function along the trajectory, which asymptotically converges to zero.

Example 6.2 (Tracking for a motion control system)
Consider an inertia $M > 0$ that is actuated using a force u, as shown in Figure 4. The position of the inertia is denoted by ξ_1 and its velocity by ξ_2. The controller force u contains a Lebesgue integrable part u_1 and an impulsive part u_2 with impulses at instants $t_i \in \mathbb{N}$. The plant is impulsive and modeled as

$$\dot{\xi} = \begin{bmatrix} \xi_2 \\ \frac{u_1}{M} \end{bmatrix} \text{ when } t \neq t_i, \; \xi^+ = \xi + \begin{bmatrix} 0 \\ \frac{u_2}{M} \end{bmatrix} \text{ when } t = t_i, \tag{69}$$

where the state ξ is assumed to be completely measured. Note that the model for the plant in (69) is time dependent and, hence, different from the model in (9). The input u will be designed, such that the state ξ tracks a reference $r = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$, shown in Figure 5 and given by

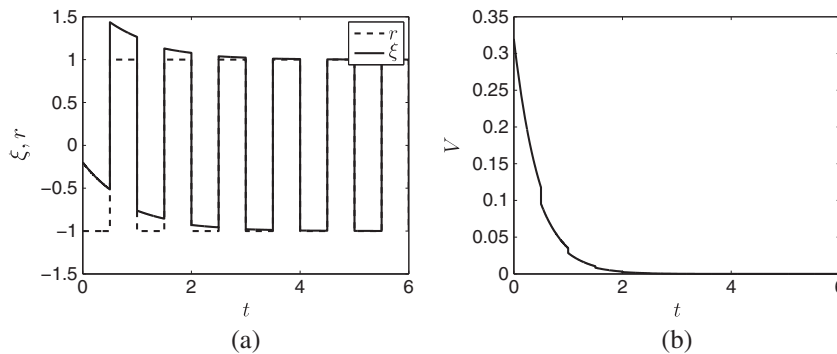(a)   (b)

Figure 3. Reference and closed-loop system trajectories for Example 6.1. The Lyapunov function along the trajectories is also shown. Parameters: $a = b = 1$ and $\lambda = 0.9$.
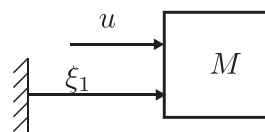

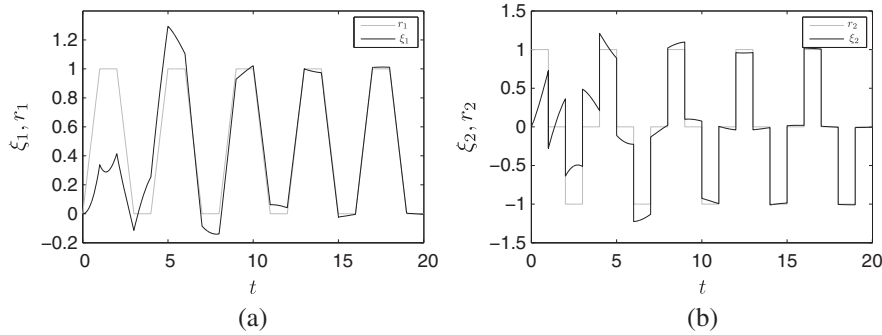Figure 4. Schematic representation of a motion control system.

Figure 5. Reference and closed-loop trajectories for Example 6.2. Parameters: $M = 1$, $\lambda_1 = 1$, and $\lambda_2 = 0.5$.

$$r = \begin{cases} \begin{bmatrix} t - j \\ 1 \end{bmatrix}, & (t, j) \in \bigcup_{k \in \mathbb{N}} [4k, 4k + 1] \times \{4k\} \\[4pt] \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & (t, j) \in \bigcup_{k \in \mathbb{N}} [4k + 1, 4k + 2] \times \{4k + 1\} \\[4pt] \begin{bmatrix} 1 - t + j \\ 0 \end{bmatrix}, & (t, j) \in \bigcup_{k \in \mathbb{N}} [4k + 2, 4k + 3] \times \{4k + 2\} \\[4pt] \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & (t, j) \in \bigcup_{k \in \mathbb{N}} [4k + 3, 4k + 4] \times \{4k + 3\} \end{cases} \tag{70}$$

The component r_2 jumps at times $(t, j) \in \mathcal{T}_r = \bigcup_{j \in \mathbb{N}} \{(j + 1, j)\}$. Such a reference trajectory can be desirable for the position of the end effector of a robot system, as it represents fast point-to-point motion. We propose the following control law to stabilize the set $\mathcal{A} = \{(\xi, \tau, k) : \xi - r(\tau, k) = 0\}$ for the given reference trajectory:

$$
\begin{aligned}
u_1 &= -\lambda_1(\xi_1 - r_1) - \lambda_2(\xi_2 - r_2), \\
u_2 &= \begin{cases} 0, & (t, j) \notin \mathcal{T}_r \\ M, & (t, j) \in \bigcup_{k \in \mathbb{N}} (4k + 3, 4k + 2) \cup (4k + 4, 4k + 3) \\ -M, & (t, j) \in \bigcup_{k \in \mathbb{N}} (4k + 1, 4k) \cup (4k + 2, 4k + 1) \end{cases},
\end{aligned}
\tag{71}
$$

where $\lambda_1, \lambda_2 > 0$.

By using the change of coordinates $e = \xi - r(\tau, k)$, the closed-loop system can be written as a hybrid system with state (e, τ, k) and dynamics given by

$$
\left. \begin{aligned}
\dot{e} &= \begin{bmatrix} 0 & 1 \\ -\frac{\lambda_1}{M} & -\frac{\lambda_2}{M} \end{bmatrix} e \\
\dot{\tau} &= 1 \\
\dot{k} &= 0
\end{aligned} \right\} \; \tau \in [t_k^r, t_{k+1}^r], \qquad
\left. \begin{aligned}
e^+ &= e \\
\tau^+ &= \tau \\
k^+ &= k + 1
\end{aligned} \right\} \; (\tau, k) \in \mathcal{T}_r. \tag{72}
$$

The feedforward signal u_2 assures that x and r experience equal jumps at the same time instances, such that e is not affected by the jumps. Furthermore, if the initial conditions are $\xi(0, 0) = r(0, 0)$, $\tau(0, 0) = k(0, 0) = 0$, then the solution satisfies $r(t, j) = \xi(t, j)$ for all $(t, j) \in \operatorname{dom} r$. Because $\lambda_1, \lambda_2 > 0$, the matrix $\begin{bmatrix} 0 & 1 \\ -\frac{\lambda_1}{M} & -\frac{\lambda_2}{M} \end{bmatrix}$ is Hurwitz. Then, we can select $V(e, \tau, k) = e^\top P e$ with $P = P^\top \succ 0$, such that

$$\langle \nabla_e V(e, \tau, k), f(e) \rangle \leqslant -V(e, \tau, k),$$

where $f(e) = \begin{bmatrix} 0 & 1 \\ -\frac{\lambda_1}{M} & -\frac{\lambda_2}{M} \end{bmatrix} e$ follows from (72). Because e is constant over jumps, we obtain

$$V(G(e, \tau, k)) - V(e, \tau, k) = 0 \qquad \forall (e, \tau, k), \tag{73}$$

where $G(e, \tau, k) = (e, \tau, k + 1)$ follows from (72). By the properties of V, there exist functions α_1, α_2 and ρ such that both (21) and (22) are satisfied. Conditions (23) and (24) hold with $\rho_1 \equiv 0$ and $\rho_2 \equiv 0$. Moreover, the hybrid time domain of each solution to the closed-loop system is such that $t_j = j$. Then, for each element (t, j) in its hybrid time domain, we have $t \geqslant j$ from where it follows that if $t + j \geqslant N$, then $t \geqslant \frac{N}{2}$. Hence, global uniform asymptotic stability of the set $\mathcal{A}$ for closed-loop system (72) follows using Corollary 4.4 (note that (72) takes the form of $\mathcal{H}_{cl}^{\star}$ with sets C_p and C_c covering the entire state space and sets D_p and D_c empty). In Figure 5, a closed-loop trajectory is shown for parameters $M = 1$, $\lambda_1 = 1$ and $\lambda_2 = 0.5$. For these system parameters, we used $P = \begin{bmatrix} 2.25 & 0.5 \\ 0.5 & 2 \end{bmatrix}$. In this example, the controller parameters are chosen to induce critically underdamped error dynamics, such that the closed-loop response shows a damped oscillation as it converges to the reference trajectory.

The next three examples illustrate the controller synthesis results presented in Section 5.

Example 6.3 (LMI-based controller synthesis without jump time restriction)
Consider system (37) with $A_p = \begin{bmatrix} 0 & 1 \\ 0.1 & 0.1 \end{bmatrix}$, $B_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $E_p = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.9 \end{bmatrix}$, and $F_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We aim to design a hybrid controller as in (39), where the controllers are synthesized using Theorem 5.2. Numerically, we find the solutions $X = \begin{bmatrix} 7.99 & -1.84 \\ -1.84 & 9.97 \end{bmatrix}$, $Y_1 = \begin{bmatrix} -9.97 & -3.33 \end{bmatrix}$, and $Y_2 = \begin{bmatrix} 0.5 & 0.25 \end{bmatrix}$ to the LMIs (42) and (43), which correspond to the controller gains $K_1 = \begin{bmatrix} -1.38 & -0.59 \end{bmatrix}$ and $K_2 = \begin{bmatrix} 0.07 & 0.04 \end{bmatrix}$. To illustrate the behavior of the closed-loop system, Figure 6 contains closed-loop trajectories of this system using a reference that is a solution to (38) with feedforward signal $u_1^r \equiv 0$ and u_2^r given in (71) for $M = 1$. Accurate tracking is obtained because the plant trajectory converges asymptotically to the reference. The Lyapunov function V is shown to decrease both during jumps, and during flow.

Example 6.4 (LMI-based controller synthesis with a minimum average interjump time condition)
In this example, the results in Theorem 5.4 are used to design a tracking controller for system (69) with $M = 1$ and design a controller to follow reference (70), that is, a solution to the hybrid system for feedforward signal $u_1^r \equiv 0$ and u_2^r given in (71). Because the feedforward signal u_2^r is nonzero once every continuous-time unit, we observe that $\{(t_j^r, j^r)\} \in \mathcal{S}_{\text{avg}}[1, 1]$. For $M = 1$, system (69) can be written as (37) with $A_p = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $E_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $F_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. To use the LMI-based controller synthesis, we select $c = 1$, $d = -1$, and apply an LMI solver to find solutions to LMIs (56) and (57). The solutions $X = \begin{bmatrix} 1.05 & -0.85 \\ -0.85 & 2.42 \end{bmatrix}$, $Y_1 = \begin{bmatrix} -1.57 & -2.29 \end{bmatrix}$, and $Y_2 = \begin{bmatrix} -0.23 & 0.61 \end{bmatrix}$ are obtained, yielding gains $K_1 = \begin{bmatrix} -3.16 & -2.05 \end{bmatrix}$ and $K_2 = \begin{bmatrix} -0.02 & 0.25 \end{bmatrix}$, for controller (39).

In Figure 7, simulations are shown for the closed-loop system. In panels (a) and (b), we observe that the plant trajectory ξ converges asymptotically to the reference r. As shown in panel (c), the Lyapunov function does increase over jumps. In fact, repetitive jumps would be unstable: the matrix $E_p + F_p K_2 = \begin{bmatrix} 1 & 0 \\ -0.02 & 1.25 \end{bmatrix}$ has eigenvalues 1 and 1.25. However, the unstable jumping behavior is compensated by stable behavior during flow, as depicted in panel (c).
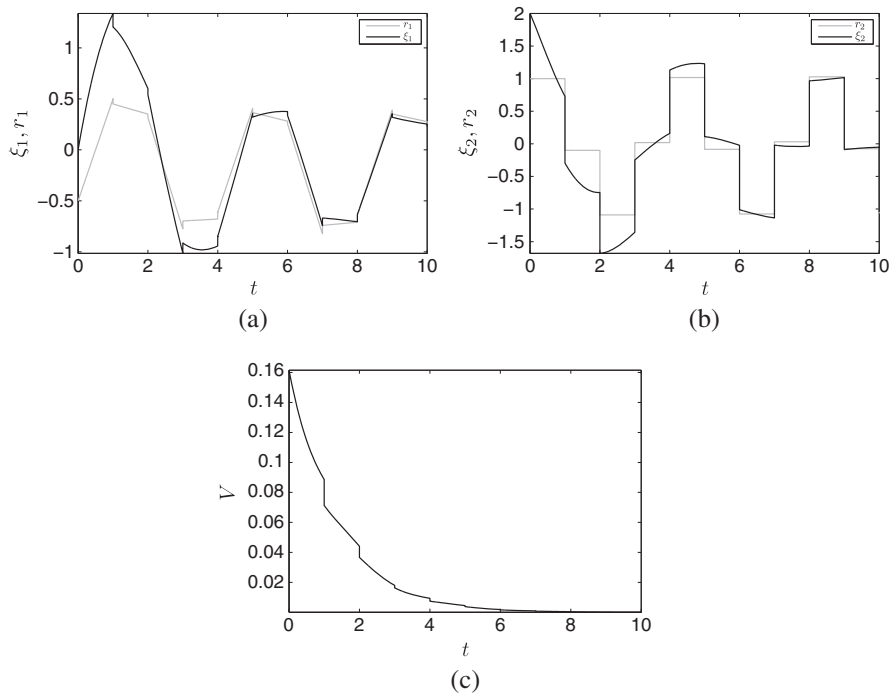
Figure 6. (a) and (b) Reference and closed-loop trajectories of system (37) and controller (39) with parameters given in Example 6.3 from initial condition $r(0,0) = (-0.5 \ 1)^\top$, $\xi(0,0) = (0 \ 2)^\top$. (c) Lyapunov function evaluated along this trajectory.
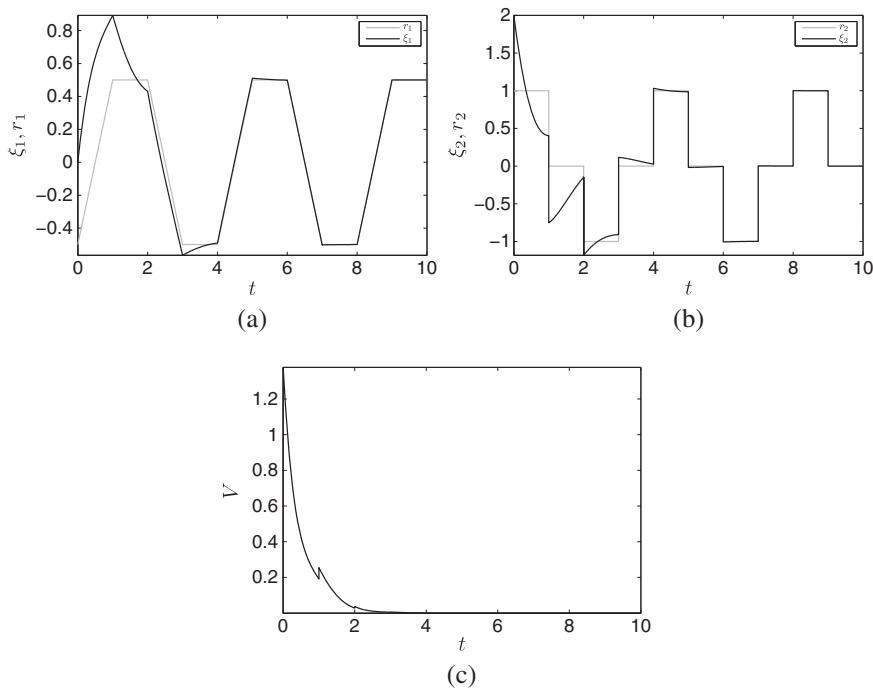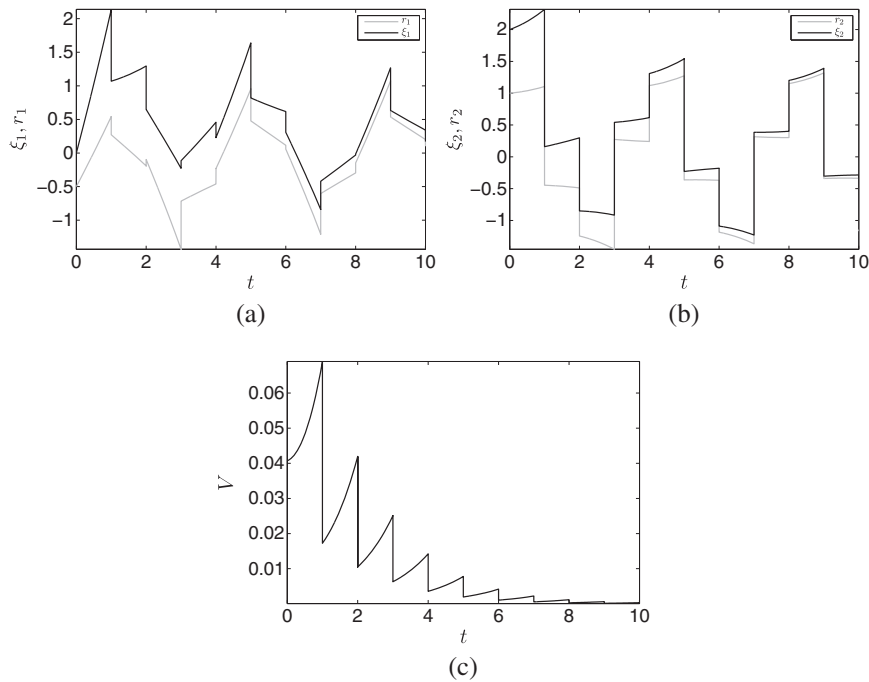

Figure 7. (a) and (b) Reference and closed-loop trajectories of (69) using the synthesized controller with parameters $K_1 = [-3.16 \ -2.05]$ and $K_2 = [-0.02 \ 0.25]$ from initial condition $r(0,0) = (-0.5 \ 1)^\top$, $\xi(0,0) = (0 \ 2)^\top$. (c) Lyapunov function evaluated along this trajectory.

Figure 8. (a) and (b) Reference and closed-loop trajectories of system (37) with controller (39) and parameters given in Section 6.5 from initial condition $r(0, 0) = (-0.5\ 1)^\top, \xi(0, 0) = (0\ 2)^\top$. (c) Lyapunov function evaluated along this trajectory.

Example 6.5 (LMI-based controller synthesis with a maximum average interjump time)
Consider system (37) with $A_p = \begin{bmatrix} 0 & 1 \\ 0.1 & 0.1 \end{bmatrix}$, $B_p = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $E_p = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$, and $F_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Note that this system has unstable flow dynamics and allows no controller input during flow. For a reference trajectory with maximum average interjump time 1, that is, $\{(t_j^r, j^r)\} \in \mathcal{S}_{\text{r-avg}}[1, 1]$, the hybrid tracking problem can be solved. Choosing $c = -1$ and $d = 1$, LMIs (56) and (57) are solved numerically to find the solutions $X = \begin{bmatrix} 61.85 & 18.20 \\ 18.20 & 25.18 \end{bmatrix}$ and $Y_2 = \begin{bmatrix} 2.67 & 1.00 \end{bmatrix}$ and arbitrary Y_1, which yields the controller gain $K_2 = \begin{bmatrix} 0.04 & 0.01 \end{bmatrix}$. The behavior for the closed-loop system is illustrated in Figure 8 for a reference that is a solution to (37) and (39) when the feedforward signal u_2^r is given in (71). As shown in panel (c) of Figure 8, the Lyapunov function does increase over flow. However, this increase is compensated by decrease of the Lyapunov function at the jump instances.

7. CONCLUSIONS

For a class of hybrid systems given in terms of hybrid inclusions, we have stated a tracking control problem for the tracking of reference signals with jumps. The proposed technique consists of embedding the reference trajectory into an extended hybrid system model and defining a set, the *tracking set*, imposing conditions on the state representing tracking of the reference trajectory. Next, Lyapunov stability tools are applied to the extended system to guarantee global asymptotic stability of the tracking set. The class of controllers should ensure that the jump times of the plant coincide with those of the given reference trajectories as, otherwise, it is difficult to obtain asymptotic convergence of the tracking error to zero (relaxation of this condition is the subject of current research and may require alternative formulations of the tracking problem). LMI-based synthesis techniques

for a class of hybrid systems with linear flow and jump maps have been proposed, and the results in the paper are illustrated by examples.

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