Output-Based Event-Triggered Control with Performance Guarantees

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Abstract—We propose an output-based event-triggered control solution for linear discrete-time systems with a performance guarantee relative to periodic time-triggered control, while reducing the communication load. The performance is expressed as an average quadratic cost and the plant is disturbed by Gaussian process and measurement noises. We establish several connections with previous works in the literature discussing, in particular, the relation to absolute and relative threshold policies. The usefulness of the results is illustrated through a numerical example.

Index Terms—Event-triggered control (ETC), output feedback, average-cost performance, optimal control.

I. INTRODUCTION

Recent research advocates that replacing periodic control and communication paradigms by event-triggered paradigms can have significant benefits in terms of reduced usage of communication, computation, and energy resources. The fundamental idea behind event-triggered control (ETC) is that transmissions should be triggered by events inferred based on available state or output information, as opposed to being triggered periodically in time.

The pioneering works [2]–[4] proposed to trigger transmissions when the norm of the state or estimation error exceeds a certain threshold (absolute triggering). In particular, [2] showed that such a policy, also known as the absolute threshold policy, can outperform periodic control in terms of a quadratic cost for the same average transmission rate for linear plants subject to Gaussian disturbances. Another influential work [5], considered a nonlinear model without disturbances and proposed to trigger transmissions in order to guarantee a decrease rate for a Lyapunov function. The resulting policy, often referred to as relative threshold policy, specifies that transmissions should occur when the norm of the error between the new and the previously sent data exceeds a weighted norm of the state. To combine the benefits of these two class of policies, in [6], mixed threshold policies were proposed, which is constructed by combining the relative and absolute threshold policies.

As an alternative, ETC policies can also be obtained from optimal control formulations taking into account the closed-loop performance

and the network usage (see, e.g., [7]–[16]). The closed-loop performance is typically defined in terms of a quadratic cost as in the celebrated linear quadratic regulator (LQR) and linear quadratic gaussian (LQG) problems. Although the optimal event-triggered controller is in general computationally hard to find for these problems, some works following this approach have proposed suboptimal event-triggered controllers with guarantees on the closed-loop performance and/or on the network usage [8], [10], [11], [17]–[19]. Interestingly, some of these suboptimal policies take the form of absolute [17] and relative [20] threshold policies, connecting well with the early works such as [2] and [5] as mentioned above.

However, such optimization-based methods typically assume that the full-state feedback is available to schedule transmissions, whereas in many applications only partial (output) information is available for feedback. In fact, there appears to be no output-based event-triggered strategy with guaranteed closed-loop quadratic performance, although output-based strategies have been proposed in the context of other design approaches for event-triggered controllers (see, e.g., [6], [21]–[24], and the references therein).

The contribution of this work is to propose a new output-feedback controller, which is guaranteed to have a performance within a constant factor of the optimal periodic control performance (with all-time communication), while reducing the communication load. Performance is defined in terms of a quadratic average cost and we consider the co-design problem in which both the control inputs and the transmission decisions are designed simultaneously. The proposed transmission policy is based on a quadratic function of the state estimate obtained by the Kalman filter, while the control input is determined by a linear function of this state estimate. Several variants are discussed.

Interestingly, one of the proposed policies takes the form of a mixed threshold policy in which the absolute threshold term is a function of the steady-state estimation error covariance and the relative part depends on the state estimate. On the other hand, in the special case when no disturbances are present, it boils down to the relative threshold policy provided in [20], which in turn is connected to the policy provided in [5].

We illustrate the usefulness of our results for the control of a system consisting of two masses and a spring using a communication network. The numerical results show that depending on the triggering mechanism, approximately up to 70% communication reduction is achieved while guaranteeing a performance within 1.1 of the optimal periodic control performance (so only 10% performance loss).

The remainder of the paper is organized as follows. In Section II, we formulate the output-feedback ETC problem. Section III explains the proposed ETC method and provides its stability and performance guarantees. Section IV presents the numerical example and Section V provides the conclusions.

A. Nomenclature

The trace of a square matrix $A \in \mathbb{R}^{n \times n}$ is denoted by $\text{Tr}(A)$. The expected value of a random vector η is denoted by $\mathbb{E}[\eta]$. For a symmetric matrix $Z \in \mathbb{R}^{n \times n}$, we write $Z \succ 0$ if Z is positive definite. For a

symmetric matrix $X \in \mathbb{R}^{n \times n}$, we use for $x \in \mathbb{R}^n$ the notation $\|x\|_X^2 := x^T X x$ and $|s|$ represents the absolute value of the scalar $s \in \mathbb{R}$. Finally, we denote the set of nonnegative integers by $\mathbb{N}_0$.

II. PROBLEM FORMULATION

We consider the linear discrete-time system

$$x_{k+1} = Ax_k + B\hat{u}_k + s_k$$

$$y_k = Cx_k + v_k, \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$, $\hat{u}_k \in \mathbb{R}^{n_u}$, and $y_k \in \mathbb{R}^{n_y}$ denote the state, the input, and the output, respectively, and s_k and v_k denote the state disturbance and the measurement noise, respectively, at discrete time $k \in \mathbb{N}_0$. We assume that $\{s_n\}_{n \in \mathbb{N}_0}$ and $\{v_n\}_{n \in \mathbb{N}_0}$ are sequences of zero-mean independent Gaussian random vectors with positive definite covariance matrices Φ_s and Φ_v, respectively. The initial state is assumed to be either a Gaussian random variable with mean $\bar{x}_0$ and covariance Θ_0 or known in which case $x_0 = \bar{x}_0$ and $\Theta_0 = 0$. Furthermore, we assume that (A, B) is controllable and (A, C) is observable.

We consider the performance measure

$$J = \limsup_{N \to \infty} \frac{1}{N} \mathbb{E} \left[\sum_{k=0}^{N-1} x_k^T Q x_k + \hat{u}_k^T R \hat{u}_k + 2 x_k^T S \hat{u}_k \right], \tag{2}$$

where Q, R, S are such that

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \succ 0. \tag{3}$$

We assume that a controller, collocated with the sensors, sends the control values to the actuators over a communication network. This controller should compute not only the control inputs, but should also decide at which times $k \in \mathbb{N}_0$ new control inputs are sent to the actuators. The setup is depicted in Fig. 1(a).

To model the occurrence of transmissions in the network, we introduce $\sigma_k \in \{0, 1\}$, $k \in \mathbb{N}_0$, as a decision variable, such that $\sigma_k = 1$ indicates that a transmission occurs at time k and $\sigma_k = 0$ otherwise. We assume that at the actuator side a standard zero-order hold device is used as a control input generator (CIG) that holds the previous value of the control action if no new control input is received at time k (i.e., when $\sigma_k = 0$). We denote the computed (and transmitted) control value at time $k \in \mathbb{N}_0$ by u_k, when a transmission occurs, and any arbitrary value otherwise. Therefore, we have

$$\hat{u}_k = \begin{cases} \hat{u}_{k-1}, & \text{if } \sigma_k = 0, \\ u_k, & \text{if } \sigma_k = 1, \end{cases} \tag{4}$$

where $\hat{u}_{-1} := 0$. When $\sigma_k = 0$, we use the notation $u_k = \emptyset$ to indicate that the value of u_k is arbitrary (and actually irrelevant). This simple hold actuation mechanism is sufficient to illustrate the main ideas of the paper. In Section III-C, we consider an alternative model-based actuation mechanism to enhance the performance of our strategy even further.

Let I_k denote the information available to the controller at time $k \in \mathbb{N}_0$, i.e.

$$I_k := \{y_0, \ldots, y_k, u_0, \ldots, u_{k-1}, \sigma_0, \ldots, \sigma_{k-1}, \hat{x}_0, \Theta_0\}.$$

A policy $\pi := (\mu_0, \mu_1, \ldots)$ is defined as a sequence of functions $\mu_k := (\mu_k^\sigma, \mu_k^u)$ that map the available information vector I_k into control actions u_k and scheduling decisions σ_k, $k \in \mathbb{N}_0$. We denote by J_π the cost (2) when policy π with

$$(\sigma_k, u_k) = \mu_k(I_k), \quad k \in \mathbb{N}_0, \tag{5}$$
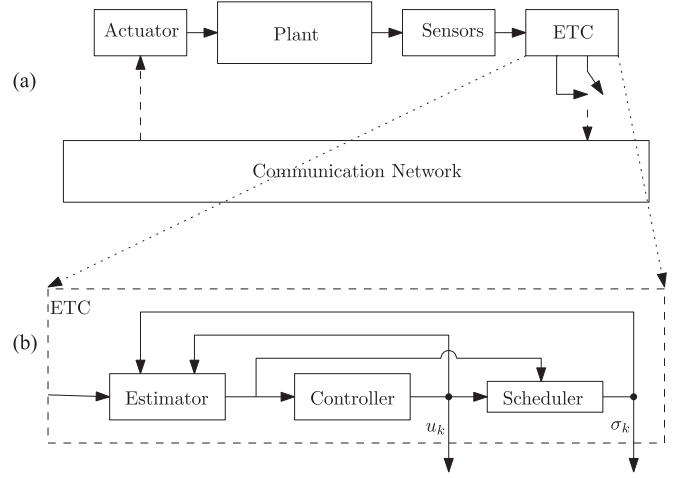

Fig. 1. Schematic of the considered structure: (a) The overall loop structure: The sensors are connected to the ETC block, which decides the transmission times and computes the control input at the transmission instant. (b) Inside the ETC: A scheduler decides based on the available sufficient statistics provided by the estimator and the event generator (16) whether or not the computed control action will be transmitted.

is used. Moreover, we denote the average transmission rate as

$$R_t = \limsup_{N \to \infty} \frac{1}{N} \mathbb{E} \left[\sum_{k=0}^{N-1} \sigma_k \right]. \tag{6}$$

Ideally, we would like to find a policy π^* that minimizes the quadratic performance index (2) as well as the average transmission rate (6). This is a multiobjective mixed-integer average cost problem, which is in general hard to solve. Instead, we propose a policy π for which the cost J_π is within a constant factor of the corresponding cost $J_{\pi_{\text{all}}}$ of periodic (all-time) control π_{all} requiring a significantly smaller average transmission rate R_t.

III. PROPOSED METHOD AND MAIN RESULTS

In Section III-A, we present the proposed event-based controller and in Section III-B we provide our main result. In Section III-C, we describe an alternative model-based actuation mechanism, and we state a similar result also for this case.

A. Proposed ETC Method

The controller structure is shown in Fig. 1(b). The estimator computes an estimate of the plant's state based on the available information I_k, i.e., $\hat{x}_k := \mathbb{E}[x_k | I_k]$ which can be obtained by iterating the Kalman filter described for $k \in \mathbb{N}_0$ by

$$\hat{x}_{k+1} = A\hat{x}_k + B\hat{u}_k + N_{k+1} z_{k+1}$$

$$z_{k+1} = y_{k+1} - C(A\hat{x}_k + B\hat{u}_k), \tag{7}$$

where $\hat{u}_k$ is defined as in (4), $\hat{x}_0 = \bar{x}_0 + N_0(y_0 - C\bar{x}_0)$ is the initial condition, and $N_s = \tilde{\Sigma}_s C^T (C\tilde{\Sigma}_s C^T + \Phi_v)^{-1}$ denotes the estimator gain, where

$$\tilde{\Sigma}_{k+1} = \Phi_s + A\tilde{\Sigma}_k A^T - A\tilde{\Sigma}_k C^T (C\tilde{\Sigma}_k C^T + \Phi_v)^{-1} C\tilde{\Sigma}_k A^T, \tag{8}$$

with initial condition $\tilde{\Sigma}_0 = \Theta_0$. Note that the estimation error covariance

$$\Sigma_k := \mathbb{E}\left[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T \,|\, I_k\right]$$

can be described by (see, e.g., [25])

$$\Sigma_k = \tilde{\Sigma}_k - \tilde{\Sigma}_k C^T (C\tilde{\Sigma}_k C^T + \Phi_v)^{-1} C\tilde{\Sigma}_k. \tag{9}$$

The controller provides the input u_k to the plant only at transmission times, i.e., at times $k \in \mathbb{N}_0$ with $\sigma_k = 1$, and is described by a linear control function of the state estimate, taking the form

$$u_k = L\hat{x}_k, \tag{10}$$

where

$$L = -(R + B^T K B)^{-1}(B^T K A + S^T)$$

$$K = Q + A^T K A - P \tag{11a}$$

$$P = (A^T K B + S)(R + B^T K B)^{-1}(B^T K A + S^T). \tag{11b}$$

Note that there exists a unique positive definite solution K to the algebraic Riccati equation (11a) and (11b) due to our assumption that (A, B) is controllable and that (3) holds (see, e.g. [25]) and that the gain L coincides with the optimal gain for a state-feedback linear quadratic regulator with an all-time transmission policy π_{all}, i.e., $(\sigma_k, u_k) = (1, L\hat{x}_k)$ for all $k \in \mathbb{N}_0$.

For the scheduler, we propose two event-triggered mechanisms (ETMs) for which we provide formal performance guarantees. The first ETM specifies that a transmission at time k occurs ($\sigma_k = 1$) if

$$\boxed{\|e_k\|_Z^2 > \|\hat{x}_k\|_Y^2 + \gamma,} \tag{12}$$

where $e_k := \hat{u}_{k-1} - L\hat{x}_k$ and $\gamma := \theta\mathrm{Tr}(Q\Sigma)$,

$$Y := \theta(Q + L^T R L + U - \epsilon I)$$

$$Z := R + (1+\theta)B^T K B + \frac{\theta}{\epsilon}(RL + S^T)(L^T R + S)$$

$$U := SL + L^T S^T,$$

and ϵ is a given constant, such that $0 < \epsilon < \lambda_{\min}(Q + L^T R L + U)$ with $\lambda_{\min}$ denoting the smallest eigenvalue of the indicated matrix. Moreover, $\theta > 0$ is a tuning knob used to express the performance with respect to the all-time transmissions policy π_{all} [see (17) below]. The second proposed ETM specifies that $\sigma_k = 1$ if

$$\boxed{\begin{bmatrix} \hat{x}_k^T & e_k^T \end{bmatrix} \Gamma \begin{bmatrix} \hat{x}_k \\ e_k \end{bmatrix} - \gamma > 0,} \tag{13}$$

with

$$\Gamma = \begin{bmatrix} -\theta(Q + L^T R L + U) & -\theta(L^T R + S) \\ -\theta(RL + S^T) & R + (1+\theta)B^T K B \end{bmatrix}. \tag{14}$$

Note that (13) has additional cross terms compared to (12), but, as we shall discuss bellow, will typically lead to less transmissions. The matrices and scalars in (12) and (13) are chosen in such a way that both ETMs will result in policies π for which we can guarantee a performance bound with respect to the optimal estimation and control corresponding to the all-time transmission policy π_{all}, i.e., $J_\pi \le (1 + \theta)J_{\pi_{\mathrm{all}}}$. It is well known [25] that the optimal policy corresponding to the all-time transmission policy π_{all} is obtained by running the Kalman filter (7) for $\sigma_k = 1$, $k \in \mathbb{N}_0$, and computing the control law (10) for every $k \in \mathbb{N}_0$. This policy has a cost

$$J_{\pi_{\mathrm{all}}} = \mathrm{Tr}(\Sigma Q) + \mathrm{Tr}\left(K(\tilde{\Sigma} - \Sigma)\right), \tag{15}$$

where $\Sigma = \lim_{s\to\infty} \Sigma_s$ and $\tilde{\Sigma} = \lim_{s\to\infty} \tilde{\Sigma}_s$.

B. Performance Bounds

The main result of the paper is presented next. We say that (1), (5) is mean square stable if $\sup_{k\in\mathbb{N}_0} \mathbb{E}[\|x_k\|^2] < \infty$ along all closed-loop trajectories.

Theorem 1: Consider system (1) with scheduling and control policy π defined as

$$(\sigma_k, u_k) = \begin{cases} (1, L\hat{x}_k), & \text{if (13) holds,} \\ (0, \emptyset), & \text{otherwise,} \end{cases} \tag{16}$$

where $\hat{x}_k$ is described by (7). Then, the system (1), (5) is mean square stable for policy π and the associated average cost satisfies

$$J_\pi \le (1 + \theta)J_{\pi_{\mathrm{all}}}. \tag{17}$$

Moreover, if we use (12) instead of (13) in (16), the same statements hold.

The parameter θ adjusts the tradeoff between the transmission rate R_t and the performance J_π. In fact, for $\theta = 0$, the transmissions are triggered when $e_k^T(R + B^T K B)e_k > 0$ which is satisfied if $e_k \ne 0$ and hence, always satisfied except for a set with zero measure. For this case, the proposed policies reduce (more or less) to the all-time transmission policy. On the other hand, to see the effect of large values of θ on the policies (12) and (13), we divide both sides of the policy inequalities by θ which lead to

$$\gamma = \mathrm{Tr}(Q\Sigma)$$

$$Y = (Q + L^T R L + U - \epsilon I)$$

$$Z = \frac{1}{\theta}(R + B^T K B) + B^T K B + \frac{1}{\epsilon}(RL + S^T)(L^T R + S)$$

$$\Gamma = \begin{bmatrix} -(Q + L^T R L + U) & -L^T R - S \\ -RL - S^T & \frac{1}{\theta}(R + B^T K B) + B^T K B \end{bmatrix}.$$

As can be seen, increasing θ (enlarging the guaranteed bound) reduces the weight of e_k, and leads to less triggering in the sense that the sets $\{(\hat{x}_k, e_k)\,|\, (12) \text{ holds}\}$ and $\{(\hat{x}_k, e_k)\,|\, (13) \text{ holds}\}$ become smaller and therefore increasing θ leads typically to less transmissions.

Before proving the theorem, several remarks are in order to establish connections between our results and existing works in the literature.

Remark 1: In the special case in which there is no process and measurement noise, and the full state is available for feedback ($C = I$), the Kalman filter estimate $\hat{x}_k$ coincides with the true state of the plant x_k, $k \in \mathbb{N}_0$. Moreover, in this case it holds that $\Sigma = 0$, $\gamma = 0$, and (12) boils down to

$$\|e_k\|_Z^2 > \|x_k\|_Y^2, \tag{18}$$

where now $e_k = \hat{u}_{k-1} - Lx_k$, $k \in \mathbb{N}_0$. This is a relative triggering policy in line with [5]. In addition (13) boils down to

$$\begin{bmatrix} x_k^T & e_k^T \end{bmatrix} \Gamma \begin{bmatrix} x_k \\ e_k \end{bmatrix} > 0, \tag{19}$$

which can be shown to be equivalent to a policy provided in [20]. In [20], a similar bound on a deterministic performance index was obtained and the connection between such policy and the policy provided in [5] was established. Note that this work extends [20] considering the incomplete information and the presence of both process and measurement noises.

Remark 2: When the process and measurement noises are not zero, the triggering law (12) resembles that of a mixed ETM [6] characterized by Z, Y, and γ. The absolute triggering part γ results from the

uncertainty about the state due to the process and measurement disturbances. One can observe that the more uncertainty there is on the estimation showing itself in Σ and resulting in a larger γ, the more reluctant the schedulers are to transmit the data in the sense that the sets $\{(\hat{x}_k, e_k)| \ (12) \ \text{holds}\}$ and $\{(\hat{x}_k, e_k)| \ (13) \ \text{holds}\}$ are smaller.

Remark 3: Let $M := \theta(L^T R + S), \theta > 0$ and $0 < \epsilon < \lambda_{\min}(Q + L^T R L + U)$, then

$$0 \le (\sqrt{\theta \epsilon} \hat{x} + \frac{1}{\sqrt{\theta \epsilon}} Me)^T (\sqrt{\theta \epsilon} \hat{x} + \frac{1}{\sqrt{\theta \epsilon}} Me)$$

$$= \theta \epsilon \hat{x}^T \hat{x} + \frac{1}{\theta \epsilon} e^T M^T Me + 2\hat{x}^T Me.$$

This inequality implies that the cross term $-2\theta \hat{x}_k^T (L^T R + S)e_k$ in (13) is upper bounded by $\theta \epsilon \hat{x}^T \hat{x} + \frac{\theta}{\epsilon} e^T (RL + S)(L^T R + S^T)e$, which in turn results in

$$\begin{bmatrix} \hat{x}_k^T & e_k^T \end{bmatrix} \Gamma \begin{bmatrix} \hat{x}_k \\ e_k \end{bmatrix} \le \begin{bmatrix} \hat{x}_k^T & e_k^T \end{bmatrix} \begin{bmatrix} -Y & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ e_k \end{bmatrix}$$

$$= \|e_k\|_Z^2 - \|\hat{x}_k\|_Y^2. \tag{20}$$

That is, if for a given value of $\hat{x}_k$ and e_k at time $k \in \mathbb{N}_0$ (13) is satisfied, meaning that a transmission is triggered at time k using ETM (13), then (12) is also satisfied, but not vice versa. Hence, (12) typically leads to more transmissions than (13).

Remark 4: Since (A, C) is observable, the discrete-time Riccati equation (8) converges to a steady-state solution $\tilde{\Sigma}$, i.e., $\tilde{\Sigma} = \lim_{s \to \infty} \tilde{\Sigma}_s$ as well. As a result, Σ_k and N_{k+1}, $k \in \mathbb{N}_0$ converge to steady-state solutions $\Sigma := \lim_{s \to \infty} \Sigma_s$ and $N = \lim_{s \to \infty} N_s$ (see, e.g., [25]).

Proof of Theorem 1: For $\xi := (x, \hat{u}) \in \mathbb{R}^{n_x + n_u}$, $u \in \mathbb{R}^{n_u}$, let $g(\xi, u, 1) = (1 + \theta)(x^T Q x + u^T R u + 2x^T S u)$ and $g(\xi, u, 0) = x^T Q x + \hat{u}^T R \hat{u} + 2x^T S \hat{u}$ and define

$$W(\hat{x}_k) := (1 + \theta)\hat{x}_k^T K \hat{x}_k.$$

Note that

$$\mathbb{E}[(x_k - \hat{x}_k)\hat{x}_k^T | I_k] = 0, \quad k \in \mathbb{N}_0, \tag{21}$$

since

$$\mathbb{E}[(x_k - \hat{x}_k)\hat{x}_k^T | I_k] = (\mathbb{E}[x_k | I_k] - \hat{x}_k)\hat{x}_k^T = 0,$$

where we used the facts that given the information I_k, $\hat{x}_k$ is deterministic and $\hat{x}_k = \mathbb{E}[x_k | I_k]$. Therefore, for a given matrix X

$$\mathbb{E}[x_k^T X x_k | I_k] = \hat{x}_k^T X \hat{x}_k + \text{Tr}(X \Sigma_k), \quad k \in \mathbb{N}_0. \tag{22}$$

Now suppose that we use the triggering policy (13) and the control policy (10) for every $k \in \mathbb{N}_0$. Then, if $\sigma_k = 1$, we have

$$\mathbb{E}[W(\hat{x}_{k+1}) + g(\xi_k, L\hat{x}_k, 1)|I_k] \tag{23}$$

$$\stackrel{(7),(22)}{=} (1 + \theta)\Big(\hat{x}_k^T (A^T K A + Q - P)\hat{x}_k + \text{Tr}(\Sigma_k Q)$$

$$+ \text{Tr}\big(K(\tilde{\Sigma}_{k+1} - \Sigma_k)\big)\Big) \tag{24}$$

$$\le W(\hat{x}_k) + (1 + \theta)\big(\text{Tr}(\Sigma Q) + \text{Tr}(K(\tilde{\Sigma} - \Sigma)) + \alpha_k\big), \tag{25}$$

where $\alpha_k = |\text{Tr}\big(K((\tilde{\Sigma}_{k+1} - \tilde{\Sigma}) - (\Sigma_k - \Sigma))\big)| + |\text{Tr}((\Sigma_k - \Sigma)Q)|$. Note that in (24) we used the fact that

$$\mathbb{E}[z_{k+1}\hat{x}_k^T | I_k] = \mathbb{E}[(CA(x_k - \hat{x}_k) + Cs_k + v_k)\hat{x}_k^T | I_k] = 0,$$

due to (21), and $z_{k+1} \sim \mathcal{N}(0, C\tilde{\Sigma}_{k+1}C^T + \Phi_v)$ for all $k \in \mathbb{N}_0$, and the equality $\text{Tr}(N_{k+1}^T K N_{k+1}(C\tilde{\Sigma}_{k+1}C^T + \Phi_v)) = \text{Tr}(K(\tilde{\Sigma}_{k+1} -$

Σ_k)). In turn, if $\sigma_k = 0$, it holds that

$$\mathbb{E}[W(\hat{x}_{k+1}) + g(\xi_k, \emptyset, 0)|I_k]$$

$$\stackrel{(7),(22)}{=} W(\hat{x}_k) + e_k^T (R + (1 + \theta)B^T K B)e_k$$

$$- \theta\big(\hat{x}_k^T (Q + L^T R L + U)\hat{x}_k + 2\hat{x}_k^T (L^T R + S)e_k$$

$$+ \text{Tr}(\Sigma Q)\big) + (1 + \theta)\big(\text{Tr}(\Sigma Q) + \text{Tr}(K(\tilde{\Sigma} - \Sigma))\big)$$

$$+ \bar{\alpha}_k \tag{26}$$

$$\stackrel{(13)}{<} W(\hat{x}_k) + (1 + \theta)\big(\text{Tr}(\Sigma Q) + \text{Tr}(K(\tilde{\Sigma} - \Sigma)) + \alpha_k\big), \tag{27}$$

where $\bar{\alpha}_k = \text{Tr}((\Sigma_k - \Sigma)Q) + (1 + \theta)\text{Tr}(K((\tilde{\Sigma}_{k+1} - \tilde{\Sigma}) - (\Sigma_k - \Sigma)))$ and (26) follows by substituting $e_k + L\hat{x}_k$ for $\hat{u}_{k-1}$. We used the same facts to obtain (24) as the ones to obtain (26). Moreover, in (27) we used that $\bar{\alpha}_k \le (1 + \theta)\alpha_k$. Therefore, by combining (25) and (27) for the proposed policy π, we have

$$\mathbb{E}[W(\hat{x}_{k+1}) + g(\xi_k, \mu^u(\hat{x}_k, e_k), \mu^\sigma(\hat{x}_k, e_k)) - W(\hat{x}_k)|I_k]$$

$$\le (1 + \theta)\Big(\text{Tr}(\Sigma Q) + \text{Tr}(K(\tilde{\Sigma} - \Sigma)) + \alpha_k\Big). \tag{28}$$

Adding (28) from $k = 0$ until $k = N - 1$, and dividing by N, and conditioning over I_0, we obtain

$$\mathbb{E}\left[\frac{1}{N} \sum_{k=0}^{N-1} \mathbb{E}[W(\hat{x}_{k+1}) - W(\hat{x}_k)|I_k]\Big|I_0\right]$$

$$+ \mathbb{E}\left[\frac{1}{N} \sum_{k=0}^{N-1} \mathbb{E}[g(\xi_k, \mu^u(\hat{x}_k, e_k), \mu^\sigma(\hat{x}_k, e_k))|I_k]\Big|I_0\right]$$

$$\le (1 + \theta)\frac{1}{N} \sum_{k=0}^{N-1} \big(\text{Tr}(\Sigma Q) + \text{Tr}(K(\tilde{\Sigma} - \Sigma)) + \alpha_k\big). \tag{29}$$

Using the tower property of conditional expectation, the first summation is given by $\frac{1}{N}(\mathbb{E}[W(\hat{x}_N) - W(\hat{x}_0)]|I_0)$ and we shall prove that $\mathbb{E}[W(\hat{x}_N)|I_0]$ is bounded for all $N \in \mathbb{N}$. If we take $\limsup_{N \to \infty}$ on both sides of (29), the left-hand side is an upper bound of J_π and the right-hand side becomes the average cost of the all-time transmission policy π_{all} multiplied by $1 + \theta$ from which we conclude (17). Note that $\limsup_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \alpha_k = 0$ as $\lim_{k \to \infty} \alpha_k = 0$ based on the Stolz–Cesaro theorem.

We now prove that $\mathbb{E}[W(\hat{x}_N)|I_0]$ is bounded and that the system is mean square stable. Since $Q \succ 0$, we have

$$g(\xi, u, i) \ge x^T Q x \ge a_1 x^T x$$

for some positive constant a_1 and all $\xi \in \mathbb{R}^{n_\xi}$, $u \in \mathbb{R}^{n_u}$, and $i \in \{0, 1\}$. Hence

$$\mathbb{E}[g(\xi_k, u_k, \sigma_k)|I_k] \stackrel{(22)}{\ge} a_1(\hat{x}_k^T \hat{x}_k + \text{Tr}(\Sigma_k)).$$

Then, from (28) we conclude that

$$\mathbb{E}[\hat{x}_{k+1}^T K \hat{x}_{k+1} - \hat{x}_k^T K \hat{x}_k | I_k] \le -\tilde{a}_1 \hat{x}_k^T \hat{x}_k + d_1, \tag{30}$$

where $\tilde{a}_1 = \frac{a_1}{1 + \theta}$ and d_1 is a positive constant, such that

$$\text{Tr}(\Sigma Q) + \text{Tr}(K(\tilde{\Sigma} - \Sigma)) + \alpha_k \le d_1 \quad \text{for all } k \in \mathbb{N}_0,$$

which exists since the estimation error covariance (9) remains bounded. Then

$$\mathbb{E}[\hat{x}_{k+1}^T K \hat{x}_{k+1} | I_k] \le c_1 \hat{x}_k^T K \hat{x}_k + d_1,$$

where $c_1 := 1 - \frac{\tilde{a}_1}{b_1}$ and b_1 is such that $K \preceq b_1 I$ and $1 - \frac{\tilde{a}_1}{b_1} < 1$. From this we conclude that (using again the tower property of conditional expectation)

$$\mathbb{E}[\hat{x}_N^T K \hat{x}_N | I_0] \leq c_1^N \hat{x}_0^T K \hat{x}_0 + \sum_{k=0}^{N-1} c_1^k d_1, \tag{31}$$

leading to the conclusion that $\mathbb{E}[\hat{x}_N^T K \hat{x}_N | I_0]$ is bounded as $N \to \infty$. Since K is positive definite, this leads to $\mathbb{E}[\hat{x}_N^T \hat{x}_N | I_0]$ being bounded as $N \to \infty$ and due to (22) so is $\mathbb{E}[x_N^T x_N | I_0]$, which shows mean square stability.

To prove that the theorem holds if the triggering mechanism (13) is replaced by (12), it suffices to observe that (28) holds also for the latter case. In fact, if (12) holds, we have $\sigma_k = 1$ and we can follow the same steps as before concluding (25). On the other hand, if (12) does not hold ($\sigma_k = 0$) from Remark 3 we know that (13) would also not hold and then the same reasoning that led to (27) can be used. The same arguments can then be applied to establish the statement of the theorem for this case. ∎

Remark 5: Note that when the state and the noise covariances are zero, we can take $d_1 = 0$ and conclude from (31) and the fact that the state is then deterministic and that the system is globally exponentially stable in a deterministic sense.

Remark 6: Parameter θ affects the closed-loop performance through c_1 in (31). As θ increases, $\tilde{a}_1$ decreases, which results in the fact that c_1 approaches 1 and hence the bound on $\mathbb{E}[\|x_k\|^2]$ will increase.

Remark 7: The method we proposed can be perceived as a rollout procedure in the context of approximate dynamic programming (see, e.g., [25]) with the base policy π_{all}. Roughly speaking, $(1 + \theta)W(\hat{x}_k)$ is employed as the approximation of the cost-to-go function in a one-step lookahead optimization, i.e.

$$J_{\pi_{\text{stp}}}(\hat{x}_k) = \min_{\sigma_k, u_k} \mathbb{E}[g(\xi_k, u_k, \sigma_k) + (1 + \theta)W(\hat{x}_{k+1}) | I_k] \tag{32}$$

and the following policy can be shown to coincide with (16)

$$\mu_\pi(\hat{x}_k) = \arg \min_{\sigma_k, u_k} \mathbb{E}[g(\xi_k, u_k, \sigma_k) + (1 + \theta)W(\hat{x}_{k+1}) | I_k], \tag{33}$$

where $J_{\pi_{\text{stp}}}(\hat{x}_k)$ is the one-step lookahead cost (see [1] for the discounted cost problem.)

C. Model-Based Actuation Mechanism

An extension to the proposed ETC structure is to consider a model-based CIG [26], instead of the holding CIG, at the actuator side. Note that the corresponding implementation would require the availability of computational power at the actuator. The considered CIG is given by

$$\hat{u}_k = L\tilde{x}_k, \tag{34}$$

where

$$\tilde{x}_k = \begin{cases} (A + BL)\tilde{x}_{k-1}, & \sigma_k = 0, \\ \hat{x}_k, & \sigma_k = 1. \end{cases} \tag{35}$$

Theorem 2: Consider system (1) with scheduling policy

$$\sigma_k = \begin{cases} 1, & \text{if (13) holds with } e_k = \tilde{x}_k - \hat{x}_k, \\ 0, & \text{otherwise} \end{cases} \tag{36}$$

and with

$$\Gamma = \begin{bmatrix} -\theta(Q + L^T R L + U) & -\theta(L^T R + S)L \\ -\theta L^T(RL + S^T) & L^T(R + (1+\theta)B^T K B)L \end{bmatrix} \tag{37}$$

and $\gamma = \theta\text{Tr}(Q\Sigma)$. Then, the system (1), (7), (34), (36) is mean square stable for the policy π_{mb} and the associated average stage cost satisfies

$$J_{\pi_{\text{mb}}} \leq (1 + \theta)J_{\pi_{\text{all}}}, \tag{38}$$

where $J_{\pi_{\text{mb}}}$ is the cost of the policy (36), (34). Moreover, if we use (12) with $Y := \theta(Q + L^T RL + U - \epsilon I)$, $Z := L^T(R + (1+\theta)B^T K B + \frac{\theta}{\epsilon}RLL^T R)L$ with $0 < \epsilon < \lambda_{\min}(Q + L^T RL + U)$ instead of (13) in (36), the same statements hold.

The proof follows similar lines of reasoning as the proof of Theorem 1 with e_k and $\hat{u}_{k-1}$ replaced by Le_k and $\tilde{u}_k := L\tilde{x}_k$, respectively. It is omitted for the sake of brevity.

Clearly, it is expected (and illustrated by the example in the next section) that a model-based CIG can guarantee similar performance while reducing the communication even further. This is expected due to better control actions when there are no transmissions.

IV. ILLUSTRATIVE EXAMPLE

We consider an output-feedback version of the numerical example considered in [8]. This example consists of two unitary masses on a frictionless surface connected by an ideal spring with spring constant k_m and moving along a one-dimensional axis. The control input is a force acting on the first mass and the outputs are the position of both masses, i.e., x_1 and x_2. The state vector is $x = \begin{bmatrix} x_1 & x_2 & v_1 & v_2 \end{bmatrix}^T$, where v_1 and v_2 are the velocities of the masses. The equations of the process are

$$\dot{x}_c = A_c x_c + B_c u_c$$
$$y_c = C x_c, \tag{39}$$

where

$$A_C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\kappa_m & \kappa_m & 0 & 0 \\ \kappa_m & -\kappa_m & 0 & 0 \end{bmatrix}, \quad B_C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \tag{40}$$

By discretizing this system (for the continuous-time dynamics see [8]) with sampling period of $t_s = 0.1$ and using $k_m = 2\pi^2$, we obtain the model (39) with

$$A = \begin{bmatrix} 0.9045 & 0.0955 & 0.0968 & 0.0032 \\ 0.0955 & 0.9045 & 0.0032 & 0.0968 \\ -1.8466 & 1.8466 & 0.9045 & 0.0955 \\ 1.8 & -1.8 & 0.0955 & 0.9045 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0049 \\ 0.0001 \\ 0.0968 \\ 0.0032 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \Phi_s = 0.01I, \quad \Phi_v = 0.01I. \tag{41}$$

A. Comparison of the Proposed Policies

We consider an average cost problem with $Q = 0.1I$, $R = 0.1$, and $S = 0$. Table I summarizes the performance of the proposed policies (12) (with $\epsilon = 0.009$) and (13) in comparison with the (periodic) all-time transmission policy π_{all} for various values of θ. In all cases, a time-varying Kalman filter is used to provide a state estimate based on the available information at each iteration. As expected, relaxing the performance requirements to $J_\pi \leq (1 + \theta)J_{\pi_{\text{all}}}$ not only reduces the network usage significantly ranging from 12% to 78% reduction (depending on θ) compared to all-time transmission policy π_{all}, but also preserves the mean-square stability of the closed-loop system.

TABLE I
COMPARISON OF AVERAGE COST AND TRANSMISSION RATES OF VARIOUS
VALUE OF θ FOR POLICIES (12) AND (13) WITH $J_{\pi_{ALL}} = 0.7335$

θ	$J_{\pi(12)}$	$J_{\pi(13)}$	R_t of (12)	R_t of (13)	$(1+\theta)J_{\pi_{all}}$
0.01	0.7397	0.7399	88%	84%	0.7408
0.1	0.7406	0.7500	80%	59%	0.8068
0.5	0.7408	0.7924	78.7%	38%	1.1002
1	0.7409	0.8292	78.5%	31%	1.4670
5	0.7409	0.9058	78.4%	23%	4.4009
10	0.7409	0.9244	78%	22%	8.0683

TABLE II
COMPARISON OF THE PERFORMANCE OF THE PROPOSED POLICIES (12)
AND (13) FOR MODEL-BASED (MB) AND ZERO-ORDER HOLD (ZOH) CIGS
WITH $\theta = 0.1$

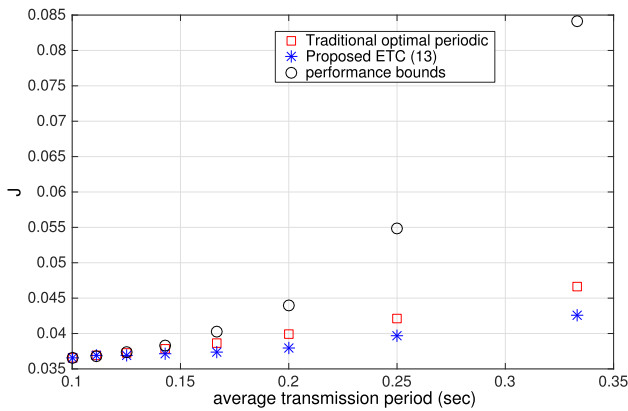| CIG	R_t of (12)	R_t of (13)	$J_{\pi(12)}$	$J_{\pi(13)}$
ZOH	80%	59%	0.7406	0.7500
MB	50%	28%	0.7419	0.7523

Fig. 2. Comparison of the ETC mechanism (13) with the traditional periodic control strategy. The values of θ used for the ETC mechanism corresponding to the points in the figure from left to right were: $\theta \in \{0, 0.005, 0.02, 0.045, 0.1, 0.2, 0.5, 1.3\}$.

Moreover, we compare four cases applying the proposed policies with $\theta = 0.1$ using different CIGs in Table II. As can be seen by using a model-based CIG an additional and significant reduction in network usage can be achieved with a similar average cost (computed through Monte Carlo simulation) compared to when a zero-order hold is used. It is also interesting to note that although choosing a large θ (e.g., $\theta = 10$) has a significant effect on the guaranteed bound, the actual cost is much less than the guaranteed bounds.

B. Comparison With Periodic Transmission Policies

Although our proposed policy has guaranteed performance bounds, a valid question is to investigate how well the policy would perform compared to the optimal periodic control policy characterized by sampling a continuous-time linear plant at equidistant time-intervals, computing and transmitting the optimal control actions to the actuators which use a holding CIG between sampling times. In Fig. 2, we compare the performances obtained with the optimal periodic control strategy and with the proposed ETC strategy (13) for several values of the average transmission period in the range [0.1, 0.33]. The parameter θ has been

tuned to obtain the desired average transmission intervals and the cost has been computed via Monte Carlo simulation of 600 realizations for 600 time units and the noise characteristics are the same as in (41). For simulation purposes, we used the discretized version of the optimal control problem specified by the original continuous-time model and the cost

$$\lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T x_c(t)^T Q_c x_c(t) + u_c(t)^T R_c u_c(t) dt \right], \quad (42)$$

where $Q_c = 0.05I$, $R_c = 0.05$. Assuming $u_c(t) = u_c(kt_s)$ for $t \in [kt_s, (k+1)t_s)$, we can obtain (1) and (2) in a similar manner as (41) for various values of t_s. The performance bounds are computed based on (17) where π_{all} corresponds to the periodic policy with $t_s = 0.1$. Note that for this simulation the corresponding values of Q, R, and S [in (2)] of the all-time transmission policy are obtained through the discretization of the cost (42) with sampling time $t_s = 0.1$. As can be seen for average transmission periods close to 0.1 (sec) the methods perform very closely. However, for larger transmission periods, the proposed strategy (13) obtains significant performance improvements over traditional periodic control with the same average transmission period, which shows the advantage of the proposed method over the traditional periodic implementation.

V. CONCLUSIONS

In this paper, we proposed an optimization-based output-feedback ETC solution for linear discrete-time systems with guaranteed performance expressed in terms of the optimal periodic (all-time) control performance, while reducing the communication load. The performance is measured by an average quadratic cost. Several connections with previous works in the literature have been established, and in particular, with the absolute, relative, and mixed threshold policies [5], [6], [27]. The usefulness of the results was illustrated through a numerical example showing that a significant (up to 72%) reduction in network usage can be achieved by only sacrificing 10% of performance compared to the optimal all-time transmission policy. Furthermore, we showed that our proposed ETC can lead to significant improvements in the performance at the same average transmission rate when compared to optimal time-triggered *periodic* controllers. For future work, we will build upon the presented results to take into account multiple channels (e.g., also including a sensor-to-actuator controller network), where the objective is to design an event-triggering mechanism for each individual channel while guaranteeing a performance bound.

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