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REVIEW

Stability analysis and controller synthesis for hybrid dynamical systems

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Wherever continuous and discrete dynamics interact, hybrid systems arise. This is especially the case in many technological systems in which logic decision-making and embedded control actions are combined with continuous physical processes. Also for many mechanical, biological, electrical and economical systems the use of hybrid models is essential to adequately describe their behaviour. To capture the evolution of these systems, mathematical models are needed that combine in one way or another the dynamics of the continuous parts of the system with the dynamics of the logic and discrete parts. These mathematical models come in all kinds of variations, but basically consist of some form of differential or difference equations on the one hand and automata or other discrete-event models on the other hand. The collection of analysis and synthesis techniques based on these models forms the research area of hybrid systems theory, which plays an important role in the multi-disciplinary design of many technological systems that surround us. This paper presents an overview from the perspective of the control community on modelling, analysis and control design for hybrid dynamical systems and surveys the major research lines in this appealing and lively research area.

Keywords: hybrid systems; piecewise affine systems; switched systems; stability analysis; controller design

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1. Introduction

Basically, hybrid systems theory studies the behaviour of dynamical systems that are modelled through mathematical formalisms that involve both continuous models, such as differential or difference equations, and discrete models, such as finite state machines or automata. The term 'hybrid systems' is used to indicate that the dynamical systems under discussion describe a mixture of heterogeneous modelling formalisms, namely continuous models as adopted in mathematical control theory and discrete models as used in computer science. In this particular context, the term was used for the first time in Witsenhausen (1966).

A textbook example of a hybrid system considers the regulation of the temperature in a house. In a simplified description, the heating system is assumed either to work at its maximum power or to be turned off. This system can operate in two modes: on and off. In each mode of operation (given by the discrete state $q \in \{\text{on, off}\}$) the changes in the temperature T can be described by a different differential equation. This is illustrated in figure 1, in which each mode corresponds to a node of a directed graph whereas the edges indicate the possible discrete state transitions. As such, this system has a hybrid state (q, T) consisting of a discrete state q, taking the discrete values on and off, and a continuous state T, taking values in the real numbers.

Clearly, the value of the discrete state q affects the evolution of the continuous state T as a different vector field is active in each mode. Vice versa, the switching between the two modes of operation is controlled by a logical device (the embedded controller) called the *thermostat* and depends on the value of the continuous state T. The mode is changed from on to off whenever the temperature T reaches the value $T_{\max} = 21$ (determined by the desired temperature, which is, say, 20°C). Vice versa, when the temperature T reaches a minimum value, $T_{\min} = 19$, the heating is switched on. This example already shows some of the main features of hybrid systems, which are as follows:

— The system has a hybrid state consisting of a discrete state q (taking a finite number of values only) and a continuous state T.
— The continuous behaviour of the system depends on the discrete state, i.e. depending on whether the mode is on or off a different dynamics $\dot{T}(t) = 30 - T(t)$ or $\dot{T}(t) = 10 - T(t)$, respectively, governs the evolution of the temperature T.
— The changes in the discrete state q are determined by the continuous state T and different conditions on T might trigger the change of the discrete state (e.g. when the discrete state is on, $T = T_{\max} = 21$ triggers the mode change, while $T = T_{\min} = 19$ triggers the change when the discrete state is off).

Although the above example is extremely simple, it already reflects the interactions between embedded controllers with a discrete character on the one hand and physical processes with a continuous character on the other hand, as are omnipresent in technological systems such as wafer scanners, electron microscopes, copiers, robotic systems, automobiles and so on. The growing complexity of technological systems demanded a deeper understanding of the complicated interaction between the discrete and continuous dynamics in the
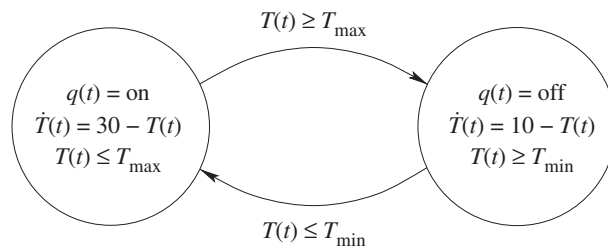
Figure 1. Model of a temperature control system.

early 1990s. As a result, the academic world started investigating the rich behaviour that is produced by these hybrid systems in the mid-1990s. This is evidenced by the start of a workshop series (Henzinger & Sastry 1998) and many special issues being published in that period (Pnueli & Sifakis 1995; Antsaklis & Nerode 1998; Morse *et al.* 1999).

In technological (man-made) systems the digital and logic (embedded control) aspects are typically brought in by design in order to control the physics and mechanics of the system. However, hybrid modelling and systems theory is not only useful within the technological domain. Many physical processes exhibiting both fast- and slow-changing behaviours can often be well described by using (simple) hybrid models. For instance, in non-smooth mechanics (Brogliato 1996), the evolution of impacting rigid bodies can be captured by hybrid models. Indeed, as the impacts occur at a much smaller time scale than the unconstrained motion, the behaviour can be described well by introducing discrete events (corresponding to impacts) and discrete (reset) actions in a smooth model. Also the vector fields defining the behaviour of the system might be different over time as they depend crucially on whether a contact is active or not. The dynamics of a robot arm moving freely in space is completely different from the situation in which it is striking the surface of an object. Other hybrid examples in mechanics include motion systems with friction models that distinguish between stick and slip modes, backlash in gears, dead zones in cog wheels, etc. Clearly, examples are not only found in the mechanical domain. Nowadays switches such as thyristors and diodes are used in electrical networks for a wide variety of applications in both power engineering and signal processing. Examples include switched-capacitor filters, modulators, analog-to-digital converters, power converters and choppers. As an example, consider diodes, which in the ideal case behave as elements with two (discrete) modes: the blocking mode and the conducting mode. Mode transitions for diodes are governed by state events, in which currents or voltages change their sign. This indicates that hybrid modelling and analysis techniques offer an attractive perspective on these switched circuits as well. Also many biological and chemical systems can often be efficiently described by hybrid models, as is clear from various papers in this special issue.

Given the wealth of applications and the urgent need for more systematic design methods for technological systems, the enormous interest in the field of hybrid systems does not come as a surprise. Various interesting results have been obtained over the years, although many problems remain unsolved at present. In this paper, we intend to provide an introduction to hybrid dynamical systems and review the main research lines from the perspective of the control community. For

more details on these research lines we will present various references for further reading. In this manner, this paper can be used as a starting point for research in the hybrid systems area.

2. A selection of hybrid modelling formats

The thermostat example has already shown some of the basic ingredients that are needed to properly model hybrid systems. A proper modelling format must involve (at least) the description of the evolution of both continuous-valued signals (temperatures, positions, velocities, currents, voltages, etc.) and discrete-valued signals (operation mode, position of a switch, alarm on or off, etc.) over time and their mutual influence. Below we will discuss various hybrid models that indeed include these characteristics.

(a) Hybrid automata

A widely used model for hybrid systems is the hybrid automaton model that combines finite-state machines/automata, as adopted in computer science, and differential equations, as used in mathematical control theory. This model was formalized for the first time by Alur *et al.* (1995) and, since then, many variations and ramifications have been proposed in the literature (e.g. Lynch *et al.* 1996; Alur & Henzinger 1997; Branicky *et al.* 1998; Lygeros *et al.* 1998; Cassandras & Lafortune 1999). There are also stochastic versions of hybrid automata, e.g. Cassandras & Lygeros (2007) and references therein. Here we present a basic deterministic version, as in figure 2, without external inputs for brevity.

Definition 2.1. A hybrid automaton H is an 8-tuple $H = (Q, X, f, \text{Init}, \text{Inv}, E, G, R)$ with

— $Q = \{q_1, \ldots, q_N\}$ is a finite set of discrete states;
— $X = \mathbb{R}^n$ is a set of continuous states;
— $f : Q \times X \to X$ is a vector field;
— $\text{Init} \subseteq Q \times X$ is a set of initial states;
— $\text{Inv} : Q \to P(X)$ describes the invariants;
— $E \subseteq Q \times Q$ is a set of edges;
— $G : E \to P(X)$ is a guard condition;
— $R : E \to P(X \times X)$ is a reset map.

Note that $P(X)$ is the power set of X, i.e. the collection of all subsets of X. Hence, the invariant Inv specifies for each mode a subset of X. Similarly, the guard condition G specifies for each mode transition or edge a subset of X. The (*hybrid*) *state* variable of the system H is given by $(q, x) \in Q \times X$.

Just to indicate briefly how this dynamical system behaves, we give a short, rather informal, description. The initial hybrid state (q_0, x_0) of 'trajectories' of a hybrid automaton must be contained in the initial set Init. From this hybrid state the continuous state x evolves according to the differential equation

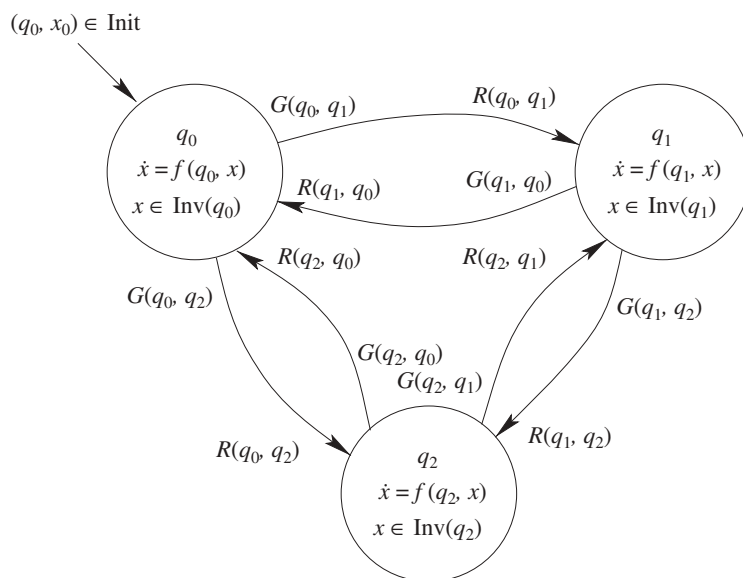$$\dot{x} = f(q_0, x) \quad \text{with } x(0) = x_0,$$

Figure 2. Schematic of a hybrid automaton with three discrete states. Each node of the directed graph represents a mode (operating point) given by a set of differential equations. The arrows indicate the possible discrete transitions that correspond to a change in the mode, which can be accompanied by resets of the continuous state variable.

while the discrete state q remains constant: $q(t) = q_0$. The continuous evolution can go on as long as x stays in $\text{Inv}(q_0)$. If at some point the continuous state x reaches the guard $G(q_0, q_1)$, we say that the transition is enabled. The discrete state may then change to q_1, and the continuous state jumps from the current value x^- to a new value x^+ that satisfies $(x^-, x^+) \in R(q_0, q_1)$.[1] After this transition, the continuous evolution resumes and the whole process is repeated.

(b) Switched systems

A quite general class of hybrid systems, although less general than hybrid automata, concerns *switched systems* given by

$$\dot{x}(t) = f_{q(t)}(x(t)), \tag{2.1}$$

where $x \in \mathbb{R}^n$ denotes the state and $q : \mathbb{R}_+ \to \{1, \ldots, N\}$ is the switching signal that determines which vector field f_q with $q \in \{1, \ldots, N\}$ is active at time $t \in \mathbb{R}_+$.

For a fixed q, switched system (2.1) describes a non-switched system, which is sometimes called a subsystem of the switched system. In case $f_q(x) = A_q x$ with $A_q \in \mathbb{R}^{n \times n}$, we speak about a *switched linear system.* The switching can depend on time only, but it can also be a function of the state $x(t)$ at time t or of an external input, and it can even have memory in it. In particular, when the switching depends only on the state variable $x(t)$ at the present time t, one speaks of *discontinuous dynamical systems* or *piecewise smooth systems.*

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[1]Sometimes the resets are given as $x^+ = r_{(q_0, q_1)}(x^-)$, which can be incorporated in the reset map by taking $R(q_0, q_1) = \{(x, r_{(q_0, q_1)}(x)) \mid x \in G(q_0, q_1)\}$.

(*c*) *Discontinuous dynamical systems and piecewise affine systems*

A typical example of a discontinuous dynamical system is the system below, which switches between two (smooth) vector fields as a result of inequalities in the state variable,

$$\dot{x}(t) = f(x(t)) = \begin{cases} f_-(x(t)) & \text{if } \phi(x(t)) \le 0, \\ f_+(x(t)) & \text{if } \phi(x(t)) \ge 0. \end{cases} \quad (2.2)$$

For this system the state space is separated into two parts by a hyper-surface defined by $\phi(x) = 0$. On one side of the surface, $\mathcal{C}_+ := \{x \in \mathbb{R}^n \mid \phi(x) > 0\}$, the dynamics $\dot{x} = f_+(x)$ holds, and on the opposite side, $\mathcal{C}_- := \{x \in \mathbb{R}^n \mid \phi(x) < 0\}$, the dynamics $\dot{x} = f_-(x)$ is valid. Hence, one can also consider this system as a *differential equation with a discontinuous right-hand side* (Filippov 1988). A distinctive difference from hybrid automata is that the state variable $x(t)$ does not have any discontinuities.

If the discontinuous dynamical system has affine dynamics in each region and the regions are polytopic (i.e. given by a finite number of linear or affine inequalities), we obtain the well-studied class of *piecewise affine* (PWA) systems (Sontag 1981, 1996), which can be considered both in a discrete-time or a continuous-time formulation. Continuous-time PWA systems are described by

$$\dot{x}(t) = A_q x(t) + B_q u(t) + f_q \quad \text{when} \quad \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{C}_q \quad (2.3)$$

for the operation modes $q \in \{1, \dots, N\}$, where $\mathcal{C}_1, \dots, \mathcal{C}_N$ are convex polyhedra in the input/state space $\mathbb{R}^{n+m}$ with non-overlapping interiors. The variables $u(t) \in \mathbb{R}^m$ and $x(t) \in \mathbb{R}^n$ denote the input and state, respectively, at time $t \in \mathbb{R}_+$. $A_q \in \mathbb{R}^{n \times n}$, $B_q \in \mathbb{R}^{n \times m}$ and $f_q \in \mathbb{R}^n$ are constant matrices for $q \in \{1, \dots, N\}$. In case $f_q = 0$ for $q \in \{1, \dots, N\}$, we speak about piecewise linear (PWL) systems.

PWA systems have been studied extensively as they form the 'simplest' extension of linear systems that can still approximate nonlinear (smooth and some non-smooth) models with arbitrary accuracy and are capable of handling some hybrid phenomena, as is shown in the discrete-time setting in Heemels *et al.* (2001).

For other classes of hybrid models such as mixed logical dynamical (MLD) models (Bemporad & Morari 1999), complementarity systems (van der Schaft & Schumacher 1996, 1998), hybrid inclusions or impulsive systems (Goebel *et al.* 2004; Goebel & Teel 2006; Haddad *et al.* 2006) timed automata (a special case of hybrid automata) (Alur & Dill 1994) and many others, the interested reader is referred to, for example, Lunze & Lamnabhi-Lagarrigue (2009) and van der Schaft & Schumacher (2000).

3. Stability analysis

The analysis of the stability property for hybrid systems has received ample attention over the years, see, for example, Liberzon (2003) or one of the surveys (Liberzon & Morse 1999; De Carlo *et al.* 2000; Shorten *et al.* 2007;

Lin & Antsaklis 2009). We will focus here mostly on the stability and stabilization of switched and discontinuous dynamical systems, and we will also indicate some results for other classes of hybrid models towards the end of this section. Loosely speaking, global asymptotic stability (GAS) of the origin 0 of switched systems as in equation (2.1) or discontinuous dynamical systems as in equation (2.2) means that for each initial state $x(0)$ the corresponding solution trajectories $x(t)$ satisfy $\lim_{t\to\infty} x(t) = 0$ (so-called *attractivity*) and that solutions starting close to the origin 0 (i.e. $x(0)$ is close to 0) remain close to 0 (so-called *Lyapunov stability*). See Khalil (1996) for more details on stability concepts.

Before we start surveying stability results for switched systems one should first formally define what is meant by the solution trajectories of the models under study, as is also reflected by the (loose) definition of GAS just given, because it requires certain properties of the solutions of the system. In addition, some well-posedness properties in the sense of existence (and maybe also uniqueness) of solutions given an initial condition might be convenient to make the stability property meaningful. Clearly, the adopted solution concept will have an influence on the stability notion. In particular, one has to be careful with certain hybrid phenomena such as Zenoness and sliding modes that do not appear in smooth systems. Zenoness indicates that there are an *infinite* number of discrete events (mode switching or resets of state variables) in a *finite* length time interval, while sliding modes occur in discontinuous dynamical systems such as equation (2.2) when the vector fields on both sides of the switching surface point towards each other (Filippov 1988). In the latter case, infinitely fast chattering between both dynamics might occur resulting in a motion ('sliding') along the switching surface. Clearly, if Zenoness and/or sliding modes occur one has to be careful with the analysis; see, for example, Liberzon (2003) and Heemels & Weiland (2008) in which this is demonstrated for the stability problem in case sliding modes are present. For overviews on solution concepts and well-posedness the reader might consult Heemels *et al.* (2003), Cortes (2008) and Lunze & Lamnabhi-Lagarrigue (2009).

Interestingly, the stability of hybrid or switched systems cannot be tackled by investigating stability of the subsystems only. This is illustrated by the following example taken from Branicky *et al.* (1998). Consider

$$\dot{x} = \begin{cases} A_1 x & \text{if } x_1 x_2 \leq 0 \\ A_2 x & \text{if } x_1 x_2 > 0 \end{cases} \quad \text{with} \quad A_1 = \begin{pmatrix} -1 & 10 \\ -100 & -1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} -1 & 100 \\ -10 & -1 \end{pmatrix}. \tag{3.1}$$

By inspection of the eigenvalues of A_1 and A_2, which for both matrices are $\lambda_{1,2} \approx -1 \pm 31.62\,\mathrm{i}$, one can see that both of the dynamics are GAS (cf. the phase portraits in figure 3), but the piecewise linear system (3.1) is not (figure 4).

This example shows that stability is a real 'hybrid problem', which requires novel stability analysis techniques beyond the large body of work on stability for continuous systems. In this survey, we will discuss two interesting questions with respect to stability of the switched system (2.1). First, we consider the case of stability under arbitrary switching and then stability for a given class of switching strategies.
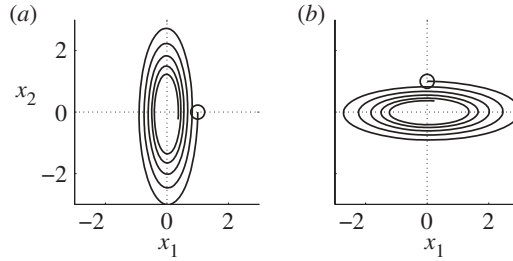
Figure 3. Behaviour of the stable linear submodels: (*a*) $\dot{x}_1 = A_1 x$ and (*b*) $\dot{x}_2 = A_2 x$, where the initial state is denoted by the circle ∘.
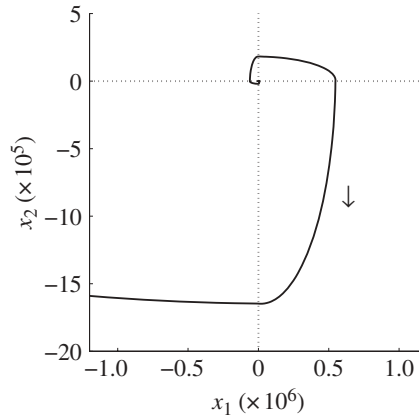

Figure 4. Unstable piecewise linear system (3.1).

(*a*) *Stability under arbitrary switching*

(i) *General switched systems*

We will say that the system (2.1) is *globally uniformly asymptotically stable* (GUAS) under arbitrary switching, if there exists a $\mathcal{KL}$ function β[2] such that for all switching signals q and any initial state $x(0)$ the corresponding solutions x of the switched system (2.1) satisfy the inequality

$$\|x(t)\| \leq \beta(\|x(0)\|, t) \quad \text{for all } t \geq 0, \tag{3.2}$$

where $\|x(t)\| := \sqrt{\sum_{i=1}^{n} x_i^2(t)}$. Note that this implies that a switched system that is GUAS under arbitrary switching has the properties $\lim_{t \to \infty} x(t) = 0$ for all solutions of the system (attractivity) and $\|x(t)\| \leq \beta(\|x(0)\|, 0)$ for all $t \geq 0$, and thus for any arbitrary small neighbourhood of 0 all trajectories can be guaranteed to stay in this neighbourhood by taking $\|x(0)\|$ sufficiently small (Lyapunov

[2]A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{KL}$ if, for each fixed $k \in \mathbb{R}_+$, $\beta(\cdot, k) \in \mathcal{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is decreasing and $\lim_{k \to \infty} \beta(s, k) = 0$.

stability). The terminology *uniform* is employed here to indicate the uniformity with respect to the switching signals: the bound (3.2) is independent of the particular choice of q.

Given a continuously differentiable (C^1) function $V : \mathbb{R}^n \to \mathbb{R}$, we will say that it is a *common Lyapunov function* for the switched system (2.1), if V is positive definite (i.e. $V(0) = 0$ and $V(x) > 0$ when $x \neq 0$) and

$$L_{f_i} V(x) := \frac{\partial V(x)}{\partial x} f_i(x) < 0 \quad \text{for all } x \neq 0 \quad \text{and all } i = 1, \ldots, N. \tag{3.3}$$

It is well established (Molchanov & Pyatnitskiy 1989; Dayawansa & Martin 1999; Liberzon & Morse 1999) that if a common Lyapunov function exists for the switched system, then the switched system is GUAS under arbitrary switching as formalized in the next theorem. We assume that the functions f_i in (2.1) are locally Lipschitz continuous in the theorems below.

Theorem 3.1. *If the switched system (2.1) has a common Lyapunov function that is radially unbounded (i.e. $V(x) \to \infty$ when $\|x\| \to \infty$), then the switched system is GUAS under arbitrary switching.*

As is also the case for smooth systems a converse Lyapunov result exists for switched systems under arbitrary switching.

Theorem 3.2 (Mancilla-Aguilar & Garcia 2000). *If the switched system (2.1) is GUAS under arbitrary switching, then it has a radially unbounded common Lyapunov function.*

Hence, GUAS under arbitrary switching of a switched system is equivalent to the existence of a radially unbounded common Lyapunov function. However, in general, it may be hard to construct such common Lyapunov functions. Nevertheless, for switched linear systems a particular class of common Lyapunov functions can be easily computed—provided there exists one.

(ii) *Switched linear systems*

Stability of switched linear systems of the form

$$\dot{x}(t) = A_{q(t)} x(t) \tag{3.4}$$

with $q(t) \in \{1, \ldots, N\}$ for all $t \in \mathbb{R}_+$ and $A_i \in \mathbb{R}^{n \times n}$, $i \in \{1, \ldots, N\}$ can also be approached by the common Lyapunov approach and, in imitation of linear systems, one can aim to use a quadratic Lyapunov function of the form $V(x) = x^\top P x$ with P a positive definite matrix.[3] If the condition (3.3) is applied for a switched linear system using a quadratic Lyapunov function $V(x) = x^\top P x$, the following result is readily obtained.

Theorem 3.3. *Suppose the set of linear matrix inequalities (LMIs)*

$$A_i^\top P + P A_i \prec 0 \quad \text{for all } i = 1, \ldots, N \tag{3.5}$$

has a solution $P \succ 0$. Then the switched linear system (3.4) is GUAS under arbitrary switching.

[3] We write $H \succ 0$ when $x^\top H x > 0$ for all $x \neq 0$ (i.e. H is positive definite) and $H \prec 0$ when $x^\top H x < 0$ for all $x \neq 0$ (i.e. H is negative definite.)

It is important to note that the above LMI conditions are only *sufficient* for GUAS under arbitrary switching, but are not *necessary.* Examples of switched linear systems exist that are GUAS under arbitrary switching, but do not allow a common *quadratic* Lyapunov function (see Dayawansa & Martin 1999). However, one can always show that if the switched linear system (3.4) is GUAS under arbitrary switching, then there is a common Lyapunov function of the form $V(x) = \max_{i=1,2,\dots,k}(l_i^\top x)^2$ for certain constant vectors l_i, $i = 1, \dots, k$ (Meilakhs 1978; Molchanov & Pyatnitskiy 1989).

Under certain additional assumptions such as pairwise commutation of $A_1, \dots, A_N$ (Narendra & Balakrishnan 1994) or all matrices $A_1, \dots, A_N$ being upper (or lower) triangularizable by the same similarity transformation (Mori *et al.* 1997; Liberzon *et al.* 1999; Agrachev & Liberzon 2001), one can actually show that a switched linear system is GUAS under arbitrary switching if and only if there exists a common quadratic Lyapunov function. Actually, under these additional assumptions, the switched linear system is GUAS under arbitrary switching if and only if all matrices $A_i, i = 1, \dots, N$ are Hurwitz (all eigenvalues having strictly negative real parts).

(*b*) *Stability for classes of switching signals*

In various situations the switching strategy is already given. As such, the switching is not arbitrary, but belongs to a specific class. Of course, one can still apply the results based on a common Lyapunov function as it shows stability for *any* switching signal, but one has to realize that such a test is conservative. In this section less conservative results will be presented for switched systems with either state-dependent or time-dependent switching. However, we will start with a general approach, as proposed by Branicky (1998).

(i) *General approach based on multiple Lyapunov functions*

A general idea for the stability analysis of switched systems (2.1) is based on multiple Lyapunov functions (Branicky 1998). The idea hinges on the fact that the underlying subsystems given by $\dot{x} = f_i(x)$, $i = 1, \dots, N$, are globally asymptotically stable (GAS). Let V_i, $i = 1, \dots, N$, be their respective Lyapunov functions. Since we assume that no common Lyapunov function exists, this means that the switched system is not GUAS under arbitrary switching and (asymptotic) stability only holds for particular classes of switching signals q (cf. theorem 3.2). Let the switching times be given by t_k, $k = 0, 1, 2, \dots$. Note that the switching times are given by the times at which the switching signal q is discontinuous. The question that arises now is for which switching signals/times GAS of the switched system can be proven.

In general, the function $t \mapsto V_{q(t)}(x(t))$ will be discontinuous. Since V_i is a Lyapunov function when submodel i ($\dot{x} = f_i(x)$) is active, V_i may increase when submodel i is not active (figure 5). Without additional conditions this can still cause $V_{q(t)}(x(t))$ to go to infinity when $t \to \infty$ as in the PWL system (3.1). Hence, additional conditions are needed if we would like to guarantee GAS. In Branicky (1998), one proposed to put conditions on the values of the ith Lyapunov function at the beginning of each time interval on which the ith submodel becomes active as formulated in the next theorem. In the formulation of the theorem we assume q to be right continuous, i.e. $\lim_{t\downarrow t_0} q(t) = q(t_0)$ for all t_0.
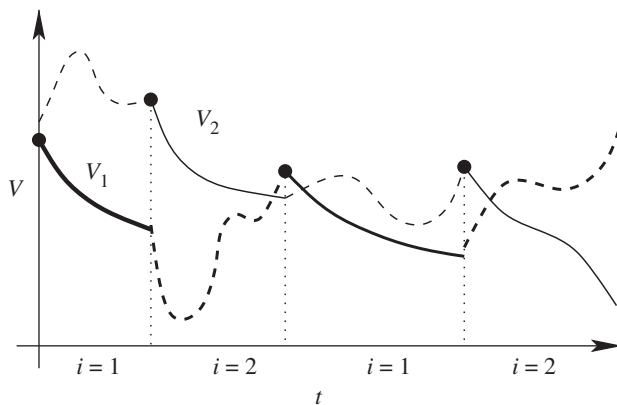
Figure 5. Multiple Lyapunov functions.

Theorem 3.4. *Consider the switched system (2.1) with all submodels $\dot{x} = f_i(x)$, $i = 1, \ldots, N$ being GAS and let V_i, $i = 1, \ldots, N$, be corresponding Lyapunov functions. Suppose that there is a class $\mathcal{K}$ function ρ with the property that for every pair of switching times (t_k, t_l), $k < l$ such that $q(t_k) = q(t_l) = i$ and $q(t_m) \neq i$ for any switching time t_m with $t_k < t_m < t_l$, we have $V_i(x(t_k)) - V_i(x(t_l)) \leq -\rho(\|x(t_l)\|)$, then the switched system is GAS for the corresponding sets of switching signals/times.*

See Branicky (1998) for a proof.

In general this theorem is difficult to apply as we need to have information on the trajectories of the system at the switching time instants. However, in the special case that the switching is induced in particular ways, life can become easier. For instance, when the switching is determined on the basis of the state variable only (e.g. as in equation (2.2)), one can try to impose continuity of the Lyapunov functions across the switching surfaces, as will be shown in the next section.

(ii) *State-dependent switching: the case of piecewise linear systems*

Consider piecewise linear systems given by

$$\dot{x} = A_i x \quad \text{when } E_i x \geq 0, \; i \in \{1, \ldots, N\}, \tag{3.6}$$

where $x \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n \times n}$ and $E_i \in \mathbb{R}^{k_i \times n}$, $i = 1, \ldots, N$. The inequality $E_i x \geq 0$ has to be interpreted componentwise and this defines a polyhedral region $\mathcal{X}_i = \{x \in \mathbb{R}^n \mid E_i x \geq 0\}$ in the state space $\mathbb{R}^n$ in which $\dot{x} = A_i x$ is active. We assume that the cells $\mathcal{X}_i$, $i \in \{1, \ldots, N\}$, form a partitioning of the state space in the sense that $\bigcup_{i=1}^{N} \mathcal{X}_i = \mathbb{R}^n$ and the cells $\mathcal{X}_i$, $i \in \{1, \ldots, N\}$, have pairwise disjoint interiors. Trajectories are assumed not to have sliding modes. Hence, we only consider solutions to the system that satisfy for almost all times $t > 0$ $\dot{x}(t) = A_i x(t)$ for some $i \in \{1, \ldots, N\}$ for which $x(t) \in \mathcal{X}_i$ holds. Moreover, we assume that there exist matrices F_i and F_j such that $F_i x = F_j x$ for all $x \in \mathcal{X}_i \cap \mathcal{X}_j$. In Johansson (2003) the construction of F_i, $i = 1, \ldots, N$, is discussed. These matrices will play an essential role in guaranteeing *continuity* of the overall Lyapunov function.

There are several relaxations possible to the construction of a *common* quadratic Lyapunov function that would imply GUAS under arbitrary switching. First of all, one can require that the derivative $L_{f_i} V(x) := (\partial V(x)/\partial x) f_i(x)$ of $V(x) = x^\top P x$ along $\dot{x} = f_i(x)$ is only negative in the region where the subsystem is active. Secondly, one can use multiple Lyapunov functions, say $V_i(x) = x^\top P_i x$ for each submodel, as in theorem 3.4. Finally, one can require that the Lyapunov function $V_i(x) = x^\top P_i x$ is only positive definite in its active region $\mathcal{X}_i$. These relaxations lead to the following theorem, as given in Johansson & Rantzer (1998).

Theorem 3.5. *If one can find symmetric matrices T, W_i and U_i with U_i and W_i having non-negative entries and such that $P_i := F_i^\top T F_i$, $i = 1, \ldots, N$ satisfy the LMIs*

$$A_i^\top P_i + P_i A_i + E_i^\top U_i E_i \prec 0, \quad i = 1, \ldots, N \tag{3.7}$$

and

$$P_i - E_i^\top W_i E_i \succ 0, \quad i = 1, \ldots, N \tag{3.8}$$

then every continuous piecewise continuously differentiable trajectory (without sliding modes) of equation (3.6) tends to zero exponentially.

As indicated at the end of §3b(i) the goal was to obtain a *continuous* Lyapunov function. Here, the Lyapunov function is chosen as $V(x) = x^\top P_i x = x^\top F_i^\top T F_i x$ when $x \in \mathcal{X}_i$. Owing to the property that was imposed on the matrices F_i, $i = 1, \ldots, N$, we have that on the boundaries of the cells, e.g. $x \in \mathcal{X}_i \cap \mathcal{X}_j$, it holds that

$$x^\top P_i x = x^\top F_i^\top T F_i x = x^\top F_j^\top T F_j x = x^\top P_j x,$$

thereby proving continuity of the Lyapunov function V. Since in each region a strict decrease of V is imposed by equation (3.7) and positive definiteness of V is guaranteed by equation (3.8), global exponential stability and thus GAS can be proved for the PWL system (3.6). The obtained conditions are only sufficient for GAS of equation (3.6), and are not necessary. Note that, in contrast with theorems 3.1 and 3.4, the hypotheses of theorem 3.5 do not require stability of the dynamics $\dot{x} = A_i x$, $i = 1, \ldots, N$.

These basic ideas can be extended in various directions. For instance, PWA systems were studied by Johansson (2003) and Rantzer & Johansson (2000). The direct inclusion of the sliding modes in the LMI-based stability analysis was considered by Heemels & Weiland (2008). For *discrete-time* switched linear systems and PWA systems the reader may consult Ferrari-Trecate *et al.* (2002*a*), Daafouz *et al.* (2002) and Johansson (2003), where stability analysis methods based on multiple or piecewise quadratic Lyapunov functions are presented. Important subtleties in the (robust) stability analysis for discrete-time discontinuous and hybrid systems were recently exposed in Lazar *et al.* (2009). Stability of piecewise polynomial systems and 'polynomial hybrid systems' (including resets of the states) using a convex programming framework for the numerical computation of Lyapunov functions (based on sum-of-squares techniques) was given in Papachristodoulou & Prajna (2009). Other approaches that allow some memory in the switching (e.g. hybrid automata with affine dynamics in each mode and without resets) are studied in, for example, Johansson (2003) and Langerak *et al.* (2003).

(iii) *Time-dependent switching: restricted dwell times*

Other classes of switching strategies can be characterized by *dwell time* restrictions on the switching signal q in the switched system (2.1). The dwell time restrictions impose limitations on how long the switched system will be in a particular mode (see Solo 1994; Hespanha & Morse 1999). For instance, one can adopt the *minimal dwell time* condition that imposes that two subsequent switching times are separated by at least $\tau_d > 0$ time units. Of course, if the system is switching between GAS subsystems and τ_d is sufficiently large (for instance, to guarantee that the norm of the state vector at a switching time is strictly smaller than the norm of the state at the previous switching time as in line with the rationale of theorem 3.4), GAS of the switched system (2.1) is obtained. In other words, if the switching between GAS subsystems is sufficiently slow, then the switched system is GAS as well. Here, we will actually consider a more general result based on *average dwell time* restrictions, which contains minimal dwell time conditions as a special case.

The concept of *average dwell time* was introduced by Hespanha & Morse (1999). To introduce this concept formally, let a switching signal $q : \mathbb{R}_+ \to \{1, \ldots, N\}$ be given. The number of discontinuities of q (switchings) on an interval (t, T) is denoted by $N_q(t, T)$. We say that q has the average dwell time τ_a if there exists a positive number N_0 such that

$$N_q(t, T) \leq N_0 + \frac{T - t}{\tau_a} \quad \text{for all} \ \ T \geq t \geq 0. \tag{3.9}$$

Note that when $N_0 = 0$ this implies that q has no discontinuities at all and, hence, there is no switching. When $N_0 = 1$ we have that q cannot switch twice on an interval of length smaller than τ_a. Hence, in the latter case we recover the case of minimal dwell time $\tau_d = \tau_a$. Generally speaking, if we discard the first N_0 switchings, then the average time between consecutive switches is at least τ_a.

Based on average dwell time restrictions, stability conditions of switched systems (2.1) are available.

Theorem 3.6. *Consider the switched system (2.1). Suppose that there exist continuously differentiable ($\mathcal{C}^1$) functions $V_i : \mathbb{R}^n \to \mathbb{R}_+$, $i = 1, \ldots, N$ and constants $0 < a < b$, $p \in \mathbb{N}$, $\mu \geq 1$, and $\lambda_0 > 0$ such that for all $i = 1, \ldots, N$*

— *$a\|x\|^p \leq V_i(x) \leq b\|x\|^p$ for all $x \in \mathbb{R}^n$;*
— *$(\partial V_i(x)/\partial x)f_i(x) \leq -\lambda_0 V_i(x)$ for all $x \in \mathbb{R}^n$;*
— *$V_i(x) \leq \mu V_j(x)$ for all $i, j \in \{1, \ldots, N\}$ and all $x \in \mathbb{R}^n$.*

Then the switched system (2.1) is globally asymptotically stable for every switching signal q with average dwell time $\tau_a > (\ln \mu)/\lambda_0$ (and N_0 arbitrary).

See Liberzon (2003) for the proof and more discussion on this interesting topic. The second condition in the above theorem states that V_i is a Lyapunov function for the ith subsystem $\dot{x} = f_i(x)$ and, hence, all subsystems are necessarily GAS. The stability under dwell time conditions of switched systems with some subsystems being unstable was studied by Liu *et al.* (2009).

(iv) *Further stability results*

In case one is interested in systems with resets of the continuous state variable one might consider, in addition to the earlier mentioned work in Papachristodoulou & Prajna (2009), the results for hybrid inclusions (sometimes also called jump-flow systems) with flow $\dot{x} \in F(x)$ if $x \in C$ and resets $x^+ \in G(x)$ if $x \in D$. The survey (Goebel *et al.* 2009) on hybrid inclusions is a good entry point to some (robust) stability theorems for such systems; also Haddad *et al.* (2006), on impulsive systems, contains various interesting results on stability, dissipativity and beyond. For more information on stability analysis of hybrid systems readers can also consult one of the surveys (Liberzon & Morse 1999; De Carlo *et al.* 2000; Shorten *et al.* 2007; Lin & Antsaklis 2009).

4. Control design

In the previous section, we have presented stability characterizations for various classes of hybrid systems. In this section, we consider the *stabilization* problem that considers the issue of how to adapt or design a given hybrid system such that it becomes stable (given the design freedom). For instance, if there are free control inputs, one could pose the question of how to choose a controller for the hybrid system, which results in a stable closed-loop system.

(*a*) *LMI-based control*

Consider the switched linear system with added control inputs $u(t) \in \mathbb{R}^m$, $t \in \mathbb{R}$,

$$\dot{x}(t) = A_{q(t)}x(t) + B_{q(t)}u(t), \quad q(t) \in I = \{1, \ldots, N\}. \tag{4.1}$$

Suppose we would like to construct a feedback controller $u(t) = K_{q(t)}x(t)$ that renders the switched linear system GUAS under arbitrary switching. This means that we have to find $K_1, \ldots, K_N$, such that

$$\dot{x}(t) = (A_{q(t)} + B_{q(t)}K_{q(t)})x(t) \tag{4.2}$$

is GUAS under arbitrary switching. Note that this set-up requires that we *know* the mode that is currently active as we schedule the feedback gains K_i accordingly. Based on the previously mentioned characterizations for GUAS under arbitrary switching, a sufficient condition is to find a common *quadratic* Lyapunov function $V(x) = x^\top P x$ for some positive definite matrix P, such that

$$(A_i + B_i K_i)^\top P + P(A_i + B_i K_i) \prec 0 \quad \text{for all } i = 1, \ldots, N \text{ and } P \succ 0. \tag{4.3}$$

These equations can be transformed into LMIs by pre-multiplying and post-multiplying by P^{-1}, yielding

$$P^{-1}(A_i + B_i K_i)^\top + (A_i + B_i K_i)P^{-1} \prec 0 \quad \text{for all } i = 1, \ldots, N \quad \text{and} \quad P^{-1} \succ 0,$$

which gives rise to the LMIs

$$Z A_i^\top + A_i Z + Y_i^\top B_i^\top + B_i Y_i \prec 0 \quad \text{for all } i = 1, \ldots, N \quad \text{and} \quad Z \succ 0$$

after the change of variables $Z := P^{-1}$ and $Y_i := K_i P^{-1}$. Once we have solved these LMIs P can be recovered as Z^{-1} and the stabilizing feedback gains are given by $K_i = Y_i Z^{-1}$, $i = 1, \ldots, N$. In case the mode is *not* known, one can try to design a *mode-independent* stabilizing feedback $u(t) = Kx(t)$ instead (by taking $K_i = K$, $i = 1, \ldots, N$ and using $Y_i = Y$, $i = 1, \ldots, N$ in the above procedure).

In the same spirit one might be interested in the stabilization of piecewise linear systems (3.6), with the control input u added just as in equation (4.1), using the ideas of theorem 3.5 together with the above transformations to obtain LMIs for the design of a PWL feedback stabilizing the system. Unfortunately, this approach results in design conditions formulated as *bilinear matrix inequalities*, which are much harder to solve than LMIs. It is widely believed that these conditions cannot be transformed into LMIs. This already indicates that for one of the simplest classes of hybrid systems, being PWL systems, the control design problem is already extremely difficult. Not many generally applicable design methodologies exist.

(b) Model predictive control

One of the few general approaches to the design of controllers for hybrid systems in the *discrete-time* setting is *model predictive control* (MPC), which is based on solving optimal control problems. One of the nice features of MPC is that it can easily include control and state constraints. The only concerns, in particular in the hybrid context, are to guarantee that the resulting control law has favourable properties such as stability and robustness and to efficiently solve and implement the optimal control problems. Below we provide a brief introduction to the general set-up of MPC, and we refer the reader for more extensive information on MPC to, for example, Mayne *et al.* (2000), Maciejowski (2002), Findeisen *et al.* (2003) and Rawlings & Mayne (2009).

(i) *MPC set-up*

Although MPC can be based on many model types, let us focus for the sake of simplicity on discrete-time PWA systems of the form

$$x_{k+1} = A_q x_k + B_q u_k + f_q \quad \text{when } x_k \in \mathcal{X}_q \qquad (4.4)$$

with the operation modes $q \in \{1, \ldots, N\}$. $A_q \in \mathbb{R}^{n \times n}$, $B_q \in \mathbb{R}^{n \times m}$ and $f_q \in \mathbb{R}^n$ are constant matrices and vectors for $q \in \{1, \ldots, N\}$ and $\mathcal{X}_1, \ldots, \mathcal{X}_N$ are convex polyhedra forming a partitioning of the set $\mathbb{X} \subseteq \mathbb{R}^n$ (i.e. $\mathbb{X} = \cup_q \mathcal{X}_q$ and the regions $\mathcal{X}_q$, $q \in \{1, \ldots, N\}$ do not overlap). The variable $x_k \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state and $u_k \in \mathbb{U} \subseteq \mathbb{R}^m$ is the control action at the discrete-time instant $k \in \mathbb{N}$. Note that model (4.4) is a discrete-time variant of equation (2.3). The sets $\mathbb{X}$ and $\mathbb{U}$ specify possible state and input constraints and it is assumed that they are compact polyhedral sets that contain the origin in their interiors. We assume for simplicity that $f_q = 0$ for all $q \in \{1, \ldots, N\}$ for which the closure of $\mathcal{X}_q$ contains the origin 0, which implies that the origin is an equilibrium state for model (4.4) with $u_k = 0$. The control goal is to stabilize this equilibrium by suitable selection of the control input u_k at each time $k \in \mathbb{N}$ exploiting information on the state variable x_k.

MPC is based on using the model (4.4) and the current information of the state x_k to predict the future behaviour of the state x over a prediction horizon $N \geq 1$, $N \in \mathbb{N}$, for choices of the future input sequence, and to

select a future input sequence that minimizes some cost criterion J. In particular, let $\mathbf{x}_k(x_k, \mathbf{u}_k) \triangleq (x_{1|k}, \ldots, x_{N|k})$ denote the state sequence generated by system (4.4) from initial state $x_{0|k} \triangleq x_k$ and by applying the input sequence $\mathbf{u}_k := (u_{0|k}, \ldots, u_{N-1|k}) \in \mathbb{U}^N$ to model (4.4), where $\mathbb{U}^N \triangleq \mathbb{U} \times \cdots \times \mathbb{U}$. The class of *admissible input sequences* defined with respect to $\mathbb{X}_T$ and state $x_k \in \mathbb{X}$ is $\mathcal{U}_N(x_k) \triangleq \{\mathbf{u}_k \in \mathbb{U}^N \mid \mathbf{x}_k(x_k, \mathbf{u}_k) \in \mathbb{X}^N, x_{N|k} \in \mathbb{X}_T\}$, where we added a terminal constraint $x_{N|k} \in \mathbb{X}_T$ to guarantee favourable properties of the closed-loop system. $\mathbb{X}_T \subseteq \mathbb{X}$ is a desired target set that contains the origin in its interior, which has to be chosen in the design. In the problem set-up below, $\mathbb{R}_+$ is defined as $\mathbb{R}_+ := [0, \infty)$.

Problem 4.1. *Let the target set $\mathbb{X}_T \subseteq \mathbb{X}$ and $N \in \mathbb{N}$ be given and let $F : \mathbb{R}^n \to \mathbb{R}_+$ with $F(0) = 0$ and $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ with $L(0, 0) = 0$ be mappings. At time $k \in \mathbb{N}$, let $x_k \in \mathbb{X}$ be given. Find $\mathbf{u}_k$ that minimizes the cost function $J(x_k, \mathbf{u}_k) := F(x_{N|k}) + \sum_{i=0}^{N-1} L(x_{i|k}, u_{i|k})$, with prediction model (4.4), over all input sequences $\mathbf{u}_k \in \mathcal{U}_N(x_k)$.*

In the MPC literature, F, L and N are called the terminal cost, the stage cost and the prediction horizon, respectively. We call an initial state $x_0 \in \mathbb{X}$ *feasible* if $\mathcal{U}_N(x_0) \neq \emptyset$. Let $\mathbb{X}_f(N) \subseteq \mathbb{X}$ denote the set of *feasible initial states*. We assume that there exists a unique optimal sequence of controls $\mathbf{u}_k^* \triangleq (u_{0|k}^*, u_{1|k}^*, \ldots, u_{N-1|k}^*)$ for problem 4.1 and any state $x_k \in \mathbb{X}_f(N)$. Then, *the MPC control law* is defined as

$$u^{\mathrm{MPC}}(x_k) \triangleq u_{0|k}^* \quad k \in \mathbb{N}. \tag{4.5}$$

Basically, the system (4.4) is controlled by the MPC law as in equation (4.5), which operates in a *receding horizon fashion* in the sense that at each time $k \in \mathbb{N}$ a new optimal control problem as in problem 4.1 (depending on x_k) is solved, providing an optimal control sequence. Only the first control move $u_{0|k}^*$ in this sequence is selected and actually implemented on the plant (4.4). At the next time $k + 1$ x_{k+1} is measured and the optimal control problem is solved again, but now for x_{k+1}. Interestingly, the above framework is quite flexible as it can be based on practically any model (as long as the solutions to the optimization problem can be computed fast enough) and can easily incorporate constraints on states and control inputs.

(ii) *Stability*

It is not always guaranteed that an MPC controller, although optimizing a performance criterion involving the state variable, indeed stabilizes the plant. For linear systems and smooth nonlinear systems several approaches are known that ensure that particular MPC set-ups are globally[4] asymptotically stable (GAS), e.g. Mayne *et al.* (2000). These are based on, for instance, putting an explicit condition such as $x_{N|k} \in \mathbb{X}_T$ and a weight F in the cost criterion on the terminal state at the end of the prediction horizon. One such example is the use of an end-point *equality* constraint of the form $x_{N|k} = 0$, i.e. $\mathbb{X}_T$ is taken to be $\{0\}$. Using such a condition in the MPC set-up, a result on attractivity (all trajectories

[4]Globally indicates here that convergence to the equilibrium state 0 is guaranteed for each trajectory starting in *any feasible* initial state.

converging to the equilibrium state) was shown in Bemporad & Morari (1999) for the case where the plant (4.4) is replaced by a so-called *mixed logical dynamical* (MLD) model, which is given by linear equations and inequalities involving real and Boolean variables. As proved in Heemels *et al.* (2001), there exist strong relationships between MLD models and PWA models, as in model (4.4) (and other model classes). Although this attractivity result is of interest and one of the first within hybrid MPC, it is rather conservative in two ways. First of all, it relies on a terminal equality constraint, which is a severe condition as it requires the property that one can regulate any initial state of interest in finite time to 0 (this property is called (null) controllability), while weaker conditions (called stabilizability in control theory) should be sufficient. Secondly, the theorem only provides attractivity and not Lyapunov stability. As such, results that relax the terminal equality constraint to milder conditions, for instance $x_{N|k} \in \mathbb{X}_T$ with $\mathbb{X}_T$ a set containing 0 in its interior, and guarantee Lyapunov stability are of interest. In Lazar *et al.* (2006) conditions are provided for asymptotic stability (including Lyapunov stability) of discrete-time PWA systems in closed loop with MPC. However, owing to the discontinuity of the corresponding closed-loop system and candidate Lyapunov function, such stability certificates may be fragile in the presence of arbitrarily small perturbations, as shown in Lazar *et al.* (2009). Therefore, guarantees of robust stability, i.e. stability in the presence of perturbations, are of importance as well and these are studied in, for example, Lazar & Heemels (2009).

(iii) *Implementation issues*

MPC is based on solving, in principle, an optimization problem as in problem 4.1 at each time $k \in \mathbb{N}$. Owing to the presence of the discrete modes this requires the solution of optimization problems involving both integers and real variables. In particular, in case of a discrete-time PWA system as in model (4.4) and a cost criterion that consists of quadratic terms this results in mixed integer quadratic programming (MIQP) problems. If a linear cost criterion is used (involving, for instance, 1- or ∞-norms) a mixed integer linear programming (MILP) problem has to be solved. Particularly convenient hybrid model formats for the formulation of hybrid MPC problems as MILP or MIQP are the earlier mentioned MLD systems (Bemporad & Morari 1999). More details on the conversion of problem 4.1 for PWA or MLD models into MILP or MIQP problems can be found in Bemporad & Morari (1999). Using the relationships between MLD and PWA systems, as in Heemels *et al.* (2001), one can use the MLD modelling format also to describe PWA systems. In Bemporad & Morari (1999) branch-and-bound methods (Fletcher & Leyffer 1998) are proposed to solve the resulting MPC optimization problems online, although other methods to solve them are also available. Unfortunately, MIQP and MILP problems are in general hard to solve (they are so-called 'NP-hard') and problems with large dimensions can easily become computationally untractable for online implementation. Exploiting the possibly available structure in the optimization problem at hand to reduce the computational burden is then of interest. As an example, in De Schutter & van den Boom (2004) implementation issues are discussed for continuous PWA systems using the continuity of the right-hand side.

An important alternative approach to solving the optimization problems online is so-called *explicit* MPC in which the optimal feedback law as in equation (4.5) is solved explicitly as a function of x_k. Indeed, if the MPC problem (4.1) gives rise to an MILP formulation, then there exists an approach based on multi-parametric linear programming in which the MPC input as in equation (4.5) can be computed *explicitly* as a PWA function of x_k. This function can be computed explicitly offline and stored in a look-up table. For the online control, one only then has to evaluate this PWA function, which has clear advantages over solving the MILP online at each discrete-time step. However, the number of regions of the PWA function also increases rapidly as the size of the MPC problem increases, which also may render the explicit MPC computationally intractable. For more information on explicit MPC we refer the reader to Bemporad *et al.* (2002*a,c*) and Borrelli (2003). It is of interest to mention that in case MPC is applied to a constrained *linear* system with quadratic costs the resulting explicit control law is also a PWA function (Bemporad *et al.* 2002*b,c*). In other words, the optimal control law for a constrained discrete-time linear system with quadratic costs is a 'hybrid' (namely PWA) state feedback. This indicates the importance of hybrid control also in the context of smooth systems.

(*c*) *Other control methods*

Given the space constraints of this survey paper, it is of course impossible to discuss all available hybrid control methodologies. Below we will provide a few additional pointers to some other approaches. In Habets *et al.* (2006) an approach is given for controller synthesis of hybrid automata with affine dynamics in each mode, polyhedral invariants and affine reset laws. This class of hybrid automata includes continuous-time PWA systems as a particular case. Based on computationally tractable conditions affine state feedback laws can be constructed that guarantee that the continuous state will leave the polyhedral invariant (region) corresponding to the affine dynamics of the present mode through a specific facet (edge) of the polyhedral region. The information of the (im)possibility is stored in a discrete-event abstraction (an automaton) that specifies which transitions can be guaranteed by proper choice of affine feedback. This discrete-event abstraction is then used to determine whether one can steer the state of the system towards specific regions avoiding possible unsafe regions. This methodology is constructive in nature and has already been used by other authors to solve (hybrid) control problems, e.g. Kloetzer & Belta (2008). Related ideas can be found in Tabuada (2009). Other ideas for the design of stabilizing switching strategies for switched linear systems (although applicable only in a rather limited setting) are provided in Wicks *et al.* (1998); see also, for example, Liberzon (2003). Based on a convex combination of the matrices A_i that is Hurwitz (if it exists), state-dependent and time-dependent switching laws can be designed, based on a quadratic Lyapunov function for the convex combination, that render the overall switched linear system GAS. Related to the above-mentioned MPC approaches, which are predominantly studied in the discrete-time context, there are also optimal control approaches for continuous-time hybrid systems (e.g. Branicky *et al.* 1998; De Schutter 2000; Rantzer & Johansson 2000; Cassandras *et al.* 2001; Hedlund & Rantzer 2002;

Xu & Antsaklis 2004 and the references therein). Also hybrid extensions are available for the celebrated Pontryagin's maximum principle, giving necessary conditions for optimality in optimal control problems (e.g. Sussmann 1999; Garavello & Piccoli 2005).

5. Other control-related topics

In this section, we briefly discuss some other interesting control theoretic topics for hybrid systems.

Observer design and state estimation for hybrid systems is also a topic of interest as the full state variable is only rarely available in practice. In this situation it might be convenient to obtain a good estimate of the (hybrid) state variable. Several papers are available on observer design for hybrid systems, especially in the context of switched and piecewise linear systems. The problem turns out to be complicated, especially when the mode of the system is not known or cannot be directly reconstructed on the basis of the available measurements. The situation becomes even more complicated when resets of the continuous state variable are present in the hybrid system at hand. Good starting points for investigating hybrid observers are, for instance, Alessandri & Coletta (2001), Bara *et al.* (2001), Juloski *et al.* (2007), Petterson (2005), Balluchi *et al.* (2002), Ferrari-Trecate *et al.* (2002*b*), van de Wouw & Pavlov (2008) and the references therein.

As observed above, the stability analysis and controller design are all model-based, which means that systematic *identification* methods to obtain appropriate hybrid models on the basis of measured data are indispensable. Some approaches for the identification of hybrid systems are described in Juloski *et al.* (2005*b*), Ferrari-Trecate *et al.* (2003), Roll *et al.* (2004), Vidal *et al.* (2003) and Bemporad *et al.* (2005), while a comparison between various identification procedures is given in Juloski *et al.* (2005*a*). The comparison paper also presents important problems that hybrid identification methods have to face.

In §3, the fundamental properties of well-posedness (existence and uniqueness of trajectories given an initial condition and external signals) was briefly mentioned. More information on this challenging subject can be found in Heemels *et al.* (2003) and Cortes (2008). Other fundamental properties of hybrid systems are *controllability, stabilizability, observability* and *detectability*, which have played a central role throughout the history of modern control theory. The complexity of characterizing controllability and stabilizability for hybrid systems has been studied in Blondel & Tsitsiklis (1999). In these papers, it was shown that even for very elementary classes of hybrid systems there is no algorithm to decide the controllability status of a given system. Hence, there is no hope to find complete conditions that characterize these properties for general hybrid systems. The best one can aim for are characterizations for some specific classes of hybrid systems. In this context controllability problems for piecewise linear systems and various related model classes have been studied by Camlibel *et al.* (2008), Lee & Arapostathis (1987), Bemporad *et al.* (2000), and Habets & van Schuppen (2001) to mention but a few. A similar story holds for observability and detectability (Bemporad *et al.* 2000; Babaali & Egerstedt 2004; Collins & Schuppen 2004; Camlibel *et al.* 2006).

6. Concluding remarks

The aim of this survey was to provide an overview of representative results in hybrid systems theory as obtained by the control community. Clearly, a survey written by computer scientists might have emphasized different aspects of hybrid systems and we refer the interested reader to the references in this survey to get a broader and more in-depth perspective on hybrid systems. In this way, we hope that this survey can serve as a good starting point to learn more about different modelling, analysis and control design techniques that are available for hybrid systems.

It is without dispute that the field of hybrid systems will grow in the near future owing to the rich application fields ranging from man-made (technological) systems to naturally occurring physical systems exhibiting hybrid behaviour, as is evidenced by the many interesting papers in this special issue. In particular, the current interest in *networked control systems* will spur on the application and development of hybrid systems theory even further in the years to come as communication networks introduce additional discrete phenomena (scheduling protocols, message loss, etc.) in the control of physical systems.

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