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Alternative frequency-domain stability criteria for discrete-time networked systems with multiple delays *

W.P.M.H. Heemels [*] **R. Merry** [*] **T. Oomen** [*]

[*] *Dept. of Mechanical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands. Email:* {m.heemels,r.j.e.merry,t.a.e.oomen}@tue.nl

Abstract: In this paper alternative frequency-domain criteria are provided for the stability of discrete-time networked control systems with time-varying delays. These criteria are in various situations less conservative than the existing frequency-domain conditions as is demonstrated by means of an example. In addition, new stability conditions are presented that allow for multiple sensor-to-controller and controller-to-actuator channels exhibiting different delay characteristics. The stability conditions are formulated in terms of the $\mathcal{H}_\infty$ norm and the structured singular value. As a result, the obtained results can be used directly for controller synthesis via standard robust control techniques.

Keywords: Delay systems, stability, discrete-time systems, $\mathcal{H}_\infty$ norm, ℓ_2 gain, structured singular value.

1. INTRODUCTION

Networked control systems (NCSs) have received considerable attention in recent years. The interest for NCSs is motivated by the many benefits they offer such as the ease of maintenance and installation, the large flexibility and the low cost. However, many issues need to be resolved before all the advantages of both wired and wireless networked control systems can be exploited. Partially, the solution lies in the improvement of the employed communication networks and protocols, resulting in increased reliability and reduction of the end-to-end latencies and packet dropouts. However, the solution can not be obtained in a cost-effective manner by only improving the communication infrastructure. Indeed, control algorithms that can deal with communication imperfections and constraints should be developed. The latter aspect is recognized widely in the control community, as evidenced by the many publications appearing recently, see e.g. the surveys Hespanha et al. (2007); Zhang et al. (2001); Tipsuwan and Chow (2003); Yang (2006); Bemporad et al. (2010).

The presence of a communication network generally induces many imperfections and constraints in the control loop, including (i) variable sampling and transmission intervals; (ii) variable communication delays; (iii) packet dropouts; (iv) communication constraints; and (v) quantization errors. To deal with these network-induced imperfections, it is crucial to understand how these imperfections influence the stability and performance of a control loop, preferably in a quantitative manner.

In this paper we investigate the stability of discrete-time NCSs that exhibit (multiple) time-varying communication delays. Dropouts can also be addressed in the framework, as these can be modeled as prolongations of the delay, see Cloosterman et al. (2010) and Remark 3.1 below. The stability of discrete-time NCSs has already been addressed using various techniques.

In Hetel et al. (2008); Song et al. (1999); Gao et al. (2004); Miani and Morassutti (2009) LMI-based stability conditions are derived assuming hard lower and upper bounds on the delays. In particular, in Hetel et al. (2008), an augmented-state model is employed in which the augmented state consists of the current state variable and the old control inputs (or states). By perceiving the resulting system as a switched linear system, LMI-based conditions for stability are derived. Also in Miani and Morassutti (2009), discrete-time NCSs with time-varying but *known* delays in the sensor-to-controller channel are analyzed and in addition, methods for controller design are proposed. Both the characterizations of (quadratic) stability and the controller design conditions are based on augmented-state models and are provided in terms of LMIs. As shown in Hetel et al. (2008), the constructed delay-dependent quadratic Lyapunov functions generalize many of the available Lyapunov-Krasovskii functionals in the literature, including the ones proposed in, e.g., Song et al. (1999); Gao et al. (2004). Recently, Lyapunov-Krasovksii type functionals are also exploited together with a delay-partitioning approach in Meng et al. (2010) to reduce the conservatism of the stability analysis.

Alternative methods to study stability of delay systems are based on using Lyapunov-Razumikhin functions, which in a discrete-time setting are reported in, for instance, Liu and Marquez (2008). In Gielen and Lazar (2009) further

relaxations regarding stability analysis using Lyapunov-Razumikhin functions are presented. Additionally, techniques for designing stabilizing receding horizon control laws are proposed.

All the methods mentioned so far assume hard lower and upper bounds on the delay variable and consider a worst-case analysis. In contrast, stability analysis based on stochastic delay characteristics is performed in, for instance, Seiler and Sengupta (2001); Zhang et al. (2005), using jump linear systems.

The main contributions of the present paper are frequency-domain characterizations for stability of discrete-time NCSs allowing multiple sensor-to-controller and controller-to-actuator channels with different delay characteristics The stability conditions for discrete-time NCSs presented in this work differ from the above mentioned results in several aspects.

Firstly, we present frequency-domain characterizations for stability based on the $\mathcal{H}_\infty$-norm and the structured singular value instead of LMI-based conditions. This is advantageous in the case that the plant is of large scale in the sense of a large number of state variables or large values for the delay parameter as this might result in stability conditions that are too complex to verify computationally. For instance, the size of the LMIs in the above mentioned references might become prohibitively large to be solved by the state-of-the-art LMI solvers. Frequency-domain characterizations suffer less from this curse of dimensionality. Note that frequency-domain characterizations of stability have been considered before in Kao and Lincoln (2004) (with a single delay). We will provide *alternative* characterizations that are less conservative in certain situations as will be demonstrated by a numerical example.

Secondly, we will allow for NCSs with different delays in different sensor-to-controller and controller-to-actuator channels. This is typically relevant for large-scale systems in which the sensors are grouped into sensor nodes that communicate independently from each other with the controller. As a result, the presented approach extends the situation considered in the aforementioned references.

Finally, instead of stochastic information as in Seiler and Sengupta (2001); Zhang et al. (2005) we adopt hard lower and upper bounds on the delays similar to Miani and Morassutti (2009); Hetel et al. (2008); Song et al. (1999); Gao et al. (2004); Meng et al. (2010); Liu and Marquez (2008). Hence, the results derived here guarantee stability irrespective of the particular probability distribution of the delays as long as the lower and upper bounds on the delays are adhered to. As our results will be formulated in terms of $\mathcal{H}_\infty$ and structured singular value (SSV) conditions (Packard and Doyle, 1993; Zhou et al., 1996; Skogestad and Postlethwaite, 2005), an interesting feature is that straightforward transformations into controller synthesis specifications are possible.

2. NOTATION AND PRELIMINARIES

If a matrix $P \in \mathbb{R}^{n \times n}$ satisfies $x^T P x > 0$ for all $x \neq 0$, then P is called positive definite and denoted by $P \succ 0$. If P satisfies $x^T P x \geq 0$ for all $x \in \mathbb{R}^n$, then P is called positive semi-definite and denoted by $P \succeq 0$. The block diagonal
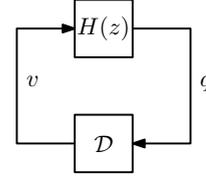

Fig. 1. Delay system.

matrix with the matrices $K^i \in \mathbb{R}^{n_i \times n_i}$, $i = 1, \ldots, L$, on the block diagonal is denoted by $\operatorname{diag}(K^1, \ldots, K^L)$.

Definition 2.1. [ℓ_2 **gain**] A discrete-time system

$$x_{k+1} = f(x_k, N_k, v_k); \ q_k = g(x_k, N_k, v_k), \qquad (1)$$

with state x_k, (disturbance) input v_k, parametric uncertainty N_k and output q_k at discrete time $k \in \mathbb{N}$ is said to have a (robust) ℓ_2 (induced) gain of γ for uncertainties in Υ, if γ is the minimal (or infimal) value of $\tilde{\gamma}$ satisfying for any input sequence $\{v_k\}_{k \in \mathbb{N}}$ with $\sum_{k=0}^\infty \|v_k\|^2 < \infty$ and any sequence $\{N_k\}_{k \in \mathbb{N}}$ of uncertainties with $N_k \in \Upsilon$, $k \in \mathbb{N}$, the inequality

$$\sum_{k=0}^\infty \|q_k\|^2 \leq \tilde{\gamma}^2 \sum_{k=0}^\infty \|v_k\|^2,$$

where $\{q_k\}_{k \in \mathbb{N}}$ is the corresponding output sequence with initial condition $x_0 = 0$.

For a linear system

$$x_{k+1} = A x_k + B v_k; \ q_k = C x_k \qquad (2)$$

the following result on ℓ_2 gains is well known, see e.g. Doyle et al. (1991); Gahinet and Apkarian (1994) for a proof.

Theorem 2.2. The following statements are equivalent:

(1) *System* (2) *has ℓ_2 gain smaller than γ.*
(2) *The $\mathcal{H}_\infty$ norm $\|H(z)\|_\infty := \sup_{z \in \mathbb{C}, |z|=1} \bar{\sigma}(H(z))$ with $H(z) = C(zI - A)^{-1}B$ is smaller than γ, where $\bar{\sigma}$ denotes the maximum singular value.*
(3) *There exist a matrix P and a $\beta \geq \frac{1}{\gamma}$ satisfying*

$$\begin{bmatrix} P - A^T P A - \beta^2 C^T C & -A^T P B \\ -B^T P A & I - B^T P B \end{bmatrix} \succeq 0 \ and \ P \succ 0.$$

$$(3)$$

3. PROBLEM FORMULATION

In this paper we are interested in the stability analysis of networked control systems (NCSs) that consists of the interconnection of a discrete-time linear system of the form

$$x_{k+1} = A x_k + B v_k; \ q_k = C x_k \qquad (4a)$$

and a communication network modeled by the time-varying delay block $\mathcal{D}$ given by

$$v_k = q_{k-N_k}. \qquad (4b)$$

Here, $x_k \in \mathbb{R}^{n_x}$ is the state and $v_k \in \mathbb{R}^{n_v}$ and $q_k \in \mathbb{R}^{n_q}$ are the interconnection variables at discrete time $k \in \mathbb{N}$. The system (4a)-(4b) can be represented as in Fig. 1 as the feedback interconnection of the discrete-time system (4a) and the varying delay block (4b). The transfer matrix of the system (4a) between v and q is denoted by $H(z) = C(zI - A)^{-1}B$. The delay N_k is time-varying and assumed to lie in the interval $[m, M] \cap \mathbb{N}$ with bounds $m, M \in \mathbb{N}$ and $0 \leq m \leq M$, i.e., $m \leq N_k \leq M$ for all $k \in \mathbb{N}$. This modeling framework encompasses

many relevant NCS architectures. For instance, in the case that the linear system (4a) models the series connection of a controller and a plant, i.e. $H(z) = P(z)C(z)$ with $C(z)$ the discrete-time transfer function matrix of the controller and $P(z)$ of the plant, the communication network induces delays in the sensor-to-controller channel. In case $H(z) = C(z)P(z)$, the delays are present in the controller-to-actuator channel. By appropriately selecting $H(z)$ also many other relevant architectures including both controller-to-actuator and sensor-to-controller delays can be incorporated in this setup.

Remark 3.1. The presented modeling framework also encompasses dropouts, since these can be considered as prolongations of the delays. Specifically, let $\bar{\delta}$ denote the bound on the maximal number of subsequent dropouts that can occur in the communication channel, then dropouts are obtained by modifying the bounds on the delay as $N_k \in [m, (\bar{\delta} + 1)M]$. The reader is referred to, e.g., Cloosterman et al. (2010), for further details.

The problem studied in this paper is to determine computationally tractable frequency-domain conditions that guarantee the global asymptotic stability of a, possible large-scale, closed-loop system given by (4) with $N_k \in [m, M] \cap \mathbb{N}$ for all $k \in \mathbb{N}$. As is argued in the introduction, the interest in frequency-domain characterizations is motivated by the fact that in various cases they provide simple and computationally tractable tests for stability. Furthermore, such frequency-domain criteria may provide useful insight for controller design. Specifically, when $H(z) = P(z)C(z)$, then the controller $C(z)$ should result in a certain upper bound on the $\mathcal{H}_\infty$ norm of $H(z)$ or $(I - H(z))^{-1}H(z)$, as will be shown in the remainder of the paper. Hereto, $\mathcal{H}_\infty$ design techniques as described in Zhou et al. (1996); Skogestad and Postlethwaite (2005) can be used to accomplish this.

4. FREQUENCY-DOMAIN CHARACTERIZATIONS

In this section, frequency domain stability characterizations are derived. Firstly, in Section 4.1, a bound on the ℓ_2 gain of a certain delay operator is derived, followed by stability characterizations of the closed-loop system (4) in Section 4.2.

4.1 ℓ_2 gain of a varying delay

To obtain frequency-domain stability characterizations, the varying delay block (4b) is recast in a state space formulation, given by

$$\zeta_{k+1} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ I_{n_q} & 0 & 0 & \dots & 0 & 0 \\ 0 & I_{n_q} & 0 & \dots & 0 & 0 \\ \vdots & & & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & I_{n_q} & 0 \end{bmatrix} \zeta_k + \begin{bmatrix} I_{n_q} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} q_k, \quad (5a)$$

$$v_k = [\Gamma_1(N_k) \dots \Gamma_M(N_k)] \zeta_k + \Gamma_0(N_k)q_k \quad (5b)$$

with $\zeta_k = (q_{k-1}^T, \dots, q_{k-M}^T)^T$ and for $i = 0, 1, \dots, M$

$$\Gamma_i(N) = \begin{cases} I_{n_q}, & \text{when } N = i, \\ 0, & \text{when } N \neq i. \end{cases} \quad (6)$$

Next, an an upper bound on the ℓ_2 gain of the system (5) is derived by using Lyapunov arguments.

Theorem 4.1. Consider the delay system (4b) *given by* $v_k = q_{k-N_k}$ *that can be represented in a state space realization as in* (5). *Let the varying* N_k, $k \in \mathbb{N}$ *be contained in* $[m, M] \cap \mathbb{N}$ *with* $m, M \in \mathbb{N}$ *and* $m \leq M$. *The ℓ_2 gain of the delay system* (5) *with delays in* $[m, M] \cap \mathbb{N}$ *is smaller than or equal to* $\sqrt{M - m + 1}$.

Proof: We will prove that the system (5) has

$$W(\zeta_k) := \sum_{i=m+1}^{M} (M-i+1)\|q_{k-i}\|^2 + \sum_{i=1}^{m} (M-m+1)\|q_{k-i}\|^2$$

as storage function for the supply rate $s(q_k, v_k) = (M - m + 1)\|q_k\|^2 - \|v_k\|^2$, i.e.

$$W(\zeta_{k+1}) - W(\zeta_k) \leq (M - m + 1)\|q_k\|^2 - \|v_k\|^2$$

for all $k \in \mathbb{N}$. This clearly implies that the ℓ_2 gain of the delay system (5) with delays in $[m, M] \cap \mathbb{N}$ is smaller than or equal to $\sqrt{M - m + 1}$.

Consider

$$W(\zeta_{k+1}) - W(\zeta_k) =$$
$$\sum_{i=m+1}^{M} (M - i + 1)\|q_{k+1-i}\|^2 + \sum_{i=1}^{m} (M - m + 1)\|q_{k+1-i}\|^2$$
$$- \sum_{j=m+1}^{M} (M - j + 1)\|q_{k-j}\|^2 - \sum_{j=1}^{m} (M - m + 1)\|q_{k-j}\|^2$$
$$\stackrel{l:=i-1}{=} \sum_{l=m}^{M-1} (M - l)\|q_{k-l}\|^2 + \sum_{l=0}^{m-1} (M - m + 1)\|q_{k-l}\|^2$$
$$- \sum_{j=m+1}^{M} (M - j + 1)\|q_{k-j}\|^2 - \sum_{j=1}^{m} (M - m + 1)\|q_{k-j}\|^2$$
$$= (M - m + 1)\|q_k\|^2 - \sum_{l=m}^{M} \|q_{k-l}\|^2$$
$$\leq (M - m + 1)\|q_k\|^2 - \|v_k\|^2,$$

where in the last inequality we used that $v_k = q_{k-N_k}$ for some $N_k \in \{m, m + 1, \dots, M\}$ (see (4b)). This completes the proof. ∎

Note that for the particular choice of the storage function in the above proof, the inequalities are tight in the sense that equalities are obtained for $q_{k-l_0} = v_k$ for some $l_0 \in [m, M] \cap \mathbb{N}$ and $q_{k-l} = 0$ for all $l \in [m, M] \cap \mathbb{N}$ and $l \neq l_0$. However, still it is unclear if there are other storage functions providing a smaller upper bound on the ℓ_2 gain than $\sqrt{M - m + 1}$. The following signal-based proof shows that this is not the case. In other words the upper bound $\sqrt{M - m + 1}$ is the true value for the ℓ_2 gain of the time-varying delay block.

Theorem 4.2. Consider the delay system (4b) *given by* $v_k = q_{k-N_k}$ *that can be represented in a state space realization as in* (5). *Let the varying* N_k, $k \in \mathbb{N}$ *be contained in* $[m, M] \cap \mathbb{N}$ *with* $m, M \in \mathbb{N}$ *and* $m \leq M$. *Then, the ℓ_2 gain of the delay system* (5) *with delays in* $[m, M] \cap \mathbb{N}$ *is equal to* $\sqrt{M - m + 1}$.

Proof: First of all we show that the ℓ_2 gain is larger than $\sqrt{M - m + 1}$ by taking the input signal

$$q_k = \begin{cases} q, & \text{when } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the input energy is given by $\sum_{k=0}^{\infty} \|q_k\|^2 = \|q\|^2$. If the varying delay acts as

$$N_k = \begin{cases} k, & \text{when } k \in [m, M], \\ \text{arbitrary}, & \text{otherwise}, \end{cases}$$

then the corresponding output v_k is given by

$$v_k = \begin{cases} q, & k \in [m, M] \\ 0, & k \notin [m, M]. \end{cases}$$

Hence, $\sum_{k=0}^{\infty} \|v_k\|^2 = (M - m + 1)\|q\|^2$ thereby showing that the ℓ_2 gain is larger than or equal to $\sqrt{M - m + 1}$. Although Theorem 4.1 can be used to show that the ℓ_2 gain is smaller than or equal to $\sqrt{M - m + 1}$, for completeness we provide also a signal-based proof of this fact. To prove that the ℓ_2 gain is smaller than or equal to $\sqrt{M - m + 1}$ using signal-based reasoning, observe that due to $v_k = q_{k-N_k}$ with $N_k \in [m, M] \cap \mathbb{N}$ we have that

$$\sum_{k=0}^{\infty} \|v_k\|^2 \leq \sum_{k=0}^{\infty} \max_{l \in [k-M, k-m]} \|q_l\|^2$$
$$\leq \sum_{k=0}^{\infty} \sum_{l=k-M}^{k-m} \|q_l\|^2 = \sum_{k=0}^{\infty} (M - m + 1)\|q_l\|^2,$$

where we used that due to initial state 0, $q_{-M} = q_{-M+1} = \ldots = q_{-1} = 0$. This completes the proof. ∎

4.2 Closed-loop stability

Based on the previous subsection, we can prove the following stability result for the closed-loop system (4). Instead of using small gain arguments, we provide a Lyapunov-based stability result, thereby constructing a Lyapunov function for the system (4).

Theorem 4.3. Consider system (4a) *with* A *Schur and* ℓ_2 *gain strictly smaller than* $\frac{1}{\sqrt{M-m+1}}$ *for* $m, M \in \mathbb{N}$ *and* $m \leq M$. *Then the NCS* (4) *with time-varying* $N_k \in [m, M] \cap \mathbb{N}$, $k \in \mathbb{N}$ *is globally asymptotically stable.*

Proof: Take the Lyapunov function candidate $V(\xi_k) = \bar{V}(x_k) + W(\zeta_k)$ with $\bar{V}(x_k) = x_k^T P x_k$ and P satisfying (3) for some $\beta^2 > M - m + 1$ and $W(\zeta_k)$ as in the proof of Theorem 4.1. Hence, using the inequality in the proof of Theorem 4.1 and it holds that $\bar{V}(x_{k+1}) - \bar{V}(x_k) \leq -\beta^2 \|q_k\|^2 + \|v_k\|^2$ for all $k \in \mathbb{N}$ due to (3), we obtain for all $k \in \mathbb{N}$

$$V(\xi_{k+1}) - V(\xi_k) \leq (M - m + 1)\|q_k\|^2 - \beta^2 \|q_k\|^2.$$

Since $\beta^2 > M - m + 1$, this gives

$$V(\xi_{k+1}) - V(\xi_k) \leq -(\beta^2 - (M - m + 1))\|q_k\|^2, \quad (7)$$

which directly proves Lyapunov stability of the closed-loop system (4). Indeed, (7) proves Lyapunov stability as $V(\xi_{k+1}) \leq V(\xi_k)$ for all $k \in \mathbb{N}$ and $c_1 \|\xi\|^2 \leq V(\xi) \leq c_2 \|\xi\|^2$ for all ξ for some $0 < c_1 \leq c_2$. To show that $\lim_{k \to \infty} \xi_k = 0$, note that by summing (7) for $k = 0, 1, \ldots, \ell$ we obtain that

$$V(\xi_{\ell+1}) - V(\xi_0) \leq -\alpha \sum_{k=0}^{\ell} \|q_k\|^2 \quad (8)$$

with $\alpha := \beta^2 - M + m - 1 > 0$ and thus $\sum_{k=0}^{\infty} \|q_k\|^2 \leq \frac{1}{\alpha} V(\xi_0)$. This implies that $q_k \to 0$ ($k \to \infty$) and due to (4b) also that $v_k \to 0$ ($k \to \infty$). Since A is Schur, this yields that $\lim_{k \to \infty} x_k = 0$ and thus $\lim_{k \to \infty} \xi_k = 0$. ∎

Interestingly, the above results show that the size of the *variation* in the delay determines the requirement on the $\mathcal{H}_\infty$ norm of the linear system, not the (absolute) size of the delay itself. Actually in case there is no variation in the delay (so $m = M$) it suffices for closed-loop stability to have A Schur and a $\mathcal{H}_\infty$ norm $\|H(z)\|_\infty$ strictly smaller than 1. The $\mathcal{H}_\infty$ norm conditions become more stringent if the delay is time-varying.

5. COMPARISON TO EARLIER FREQUENCY-DOMAIN CHARACTERIZATIONS

Earlier frequency-domain characterizations for discrete-time delay systems as in (4) are given in Kao and Lincoln (2004). In particular, in Kao and Lincoln (2004) it is shown that if for all $z \in \mathbb{C}$ with $|z| = 1$ it holds that $|\frac{H(z)}{1-H(z)}| < \frac{1}{M|z-1|}$, then the single-input single-output (SISO) system depicted in Fig. 1 is stable for $N_k \in [0, M] \cap \mathbb{N}$. These conditions are in various situations more conservative than our derived $\mathcal{H}_\infty$-based conditions, as is exemplified next.

Example 5.1. Based on Theorem 4.3 for $H(z) = \frac{\rho}{z}$ with $0 \leq \rho < 1$, stability is guaranteed for $N_k \in [0, \lceil \frac{1}{\rho^2} \rceil - 2] \cap \mathbb{N}$ as the $\mathcal{H}_\infty$ norm of $H(z)$ is equal to ρ, where $\lceil r \rceil := \min\{n \in \mathbb{N} \mid n \geq r\}$. The condition in Kao and Lincoln (2004) only guarantees stability for $N_k \in [0, \lceil \frac{1-\rho}{2\rho} \rceil - 1] \cap \mathbb{N}$. Clearly, $\frac{1}{\rho^2} > \frac{1-\rho}{2\rho}$ for $\rho \in (0, 1)$. In case that $\rho = 0.1$ the newly proposed method (with $m = 0$) guarantees stability for $N_k \in \{0, 1, \ldots, 98\}$, while the method in Kao and Lincoln (2004) guarantees stability for $N_k \in \{0, 1, 2, 3, 4\}$. When $\rho = 0.01$ the new method gives $M = 9998$, while the one in Kao and Lincoln (2004) guarantees stability only for $M = 49$.

The example reveals that in the case that $H(z)$ represents an asymptotically stable system with a bounded ℓ_2 gain/$\mathcal{H}_\infty$ norm, then the novel conditions presented in this paper result in less conservative conditions. At a first glance, it appears that the asymptotic stability of the system is a restrictive condition, e.g., when compared to the results in Kao and Lincoln (2004). In the next section, it is shown that the asymptotic stability requirement on the system can be relaxed, provided that the system is stable under unit feedback.

6. FREQUENCY-DOMAIN CHARACTERIZATIONS FOR OPEN-LOOP UNSTABLE SYSTEMS

In the case that the open-loop system is unstable, then the stability conditions that are derived in the previous sections cannot be applied directly. In this section, an approach is investigated to relax the open-loop stability requirement. Specifically, by applying a loop transformation, see Kao and Lincoln (2004) for details, the system depicted in Fig. 1 can be recast as in Fig. 2. This shows that the system (4) can be represented as the interconnection of a system Σ with transfer matrix $(I - H(z))^{-1}H(z)$ between the newly introduced signals w and z, and a linear time-varying uncertainty block given by

$$w_k = z_{k-N_k} - z_k \quad (9)$$

with $N_k \in [m, M] \cap \mathbb{N}$. Note that a state space realization for $(I - H(z))^{-1}H(z)$ can easily be obtained from (4a) given by
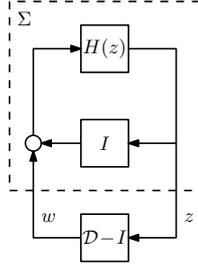
Fig. 2. Loop transformation of the delay system.

$$x_{k+1} = (A + BC)x_k + Bw_k; \ z_k = Cx_k. \qquad (10)$$

Note that in this way we can also handle open-loop unstable systems.

Using a similar reasoning as in Section 4 we are deriving now an upper bound on the ℓ_2 gain of (9).

Theorem 6.1. Consider the system (9). *Let the varying* N_k, $k \in \mathbb{N}$ *be contained in* $[m, M] \cap \mathbb{N}$ *with* $m, M \in \mathbb{N}$ *and* $m \leq M$. *The* ℓ_2 *gain of the system* (9) *with delays in* $[m, M] \cap \mathbb{N}$ *is smaller than or equal to* $1 + \sqrt{M - m + 1}$.

Proof: We provide a signal-based proof here. Using the triangle inequality for ℓ_2 norms we obtain

$$\sqrt{\sum_{k=0}^{\infty} \|w_k\|^2} \leq \sqrt{\sum_{k=0}^{\infty} \|z_{k-N_k}\|^2} + \sqrt{\sum_{k=0}^{\infty} \|z_k\|^2}$$

$$\leq (\sqrt{M - m + 1} + 1)\sqrt{\sum_{k=0}^{\infty} \|z_k\|^2},$$

where we used Theorem 4.1 in the latter inequality. This completes the proof. ∎

Hence, using small gain arguments we can now derive the following result.

Theorem 6.2. System (4) *with time-varying* $N_k \in [m, M] \cap \mathbb{N}$, $k \in \mathbb{N}$, *for* $m, M \in \mathbb{N}$ *and* $m \leq M$, *is globally asymptotically stable, if* $A + BC$ *Schur and the* ℓ_2 *gain of the system* (10), *i.e.,* $\|(I - H(z))^{-1}H(z)\|_{\infty} = \sup_{z \in \mathbb{C}, |z|=1} \bar{\sigma}((I - H(z))^{-1}H(z))$ *is strictly smaller than* $\frac{1}{1+\sqrt{M-m+1}}$.

Remark 6.3. In Kao and Lincoln (2004) one additionally used a loop transformation including $\frac{z}{z-1}$ and its inverse $\frac{z-1}{z}$, which is not necessary and results in the different stability characterization for SISO systems $|\frac{H(z)}{1-H(z)}| < \frac{1}{M|z-1|}$ for all $z \in \mathbb{C}$ with $|z| = 1$, while we obtain $|\frac{H(z)}{1-H(z)}| < \frac{1}{1+\sqrt{M+1}}$ for all $z \in \mathbb{C}$ with $|z| = 1$ (in case $m = 0$). Especially, for large delays (M large) our results might be less conservative.

7. NCS WITH MULTIPLE DELAYS

All the previous results considered the case of one single, possibly multiple input multiple output, delay block. For large-scale systems in which multiple sensor nodes communicate separately with the controller this assumption commonly is not satisfied. In this section, generalizations of the results in the previous sections are presented that allow for different delay blocks between the inputs and outputs of (4a). Hereto, consider the situation in which the
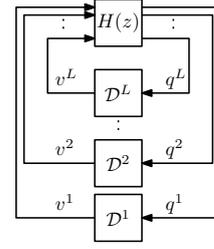

Fig. 3. Multiple delay system.

outputs of the plant are grouped into sensor nodes. These sensor nodes are generally connected to the controller using different wired or wireless communication media. As a result, the corresponding information arrives after different delays, as is represented by the block diagram in Fig. 3.

To formalize the above situation, consider the dynamics

$$x_{k+1} = Ax_k + Bv_k; \ q_k = Cx_k \qquad (11)$$

with $H(z) := C(zI - A)^{-1}B$. As mentioned in Section 3, in case the delays are present in the sensor-to-controller channels $H(z)$ is equal to $P(z)C(z)$ with $C(z)$ the transfer matrix of the controller and $P(z)$ of the open-loop plant. Other network configurations, e.g., including delays acting between controller and actuators, can straightforwardly be included in the presented modeling framework. The outputs are grouped into L sensor nodes and the outputs attributed to node $i \in \{1, \ldots, L\}$ are denoted by $q^i \in \mathbb{R}^{n_i}$ and the corresponding delayed outputs by $v^i \in \mathbb{R}^{n_i}$. Hence, node i corresponds to n_i, $i = 1, \ldots, L$, outputs. In addition, the measured output signal q_k at time k can be written as $q_k = ((q_k^1)^T \ldots (q_k^L)^T)^T$, and input v_k as $v_k = ((v_k^1)^T \ldots (v_k^L)^T)^T$. The outputs q_k^i arrive at the controller after a delay $N_k^i \in [m_i, M_i] \cap \mathbb{N}$, $i \in \{1, \ldots, L\}$, i.e.

$$v_k^i = \mathcal{D}^i q := q_{k-N_k^i}^i, \qquad (12)$$

where $N_k^i \in [m_i, M_i] \cap \mathbb{N}$ for all $k \in \mathbb{N}$ with $m_i, M_i \in \mathbb{N}$ and $m_i \leq M_i$, $i \in \{1, \ldots, L\}$. Hence, each sensor node has its own delay characteristics. For instance, more remote nodes or nodes using slower communication media typically correspond to larger values for m_i and M_i. The delays in (12), $i \in \{1, \ldots, L\}$, are united in one (multiple) delay block $\mathcal{D} := \text{diag}(\mathcal{D}^1, \ldots, \mathcal{D}^L)$.

The ℓ_2 characterizations derived for the delay blocks, as derived in the previous sections, in conjunction with results of the structured singular value, offer an elegant and rather straightforward method to perform a stability analysis. To elaborate further on these ideas, consider the set $\mathcal{K}$ of invertible matrices that have a block structure compatible with $\mathcal{D}$ in the sense that any $K \in \mathcal{K}$ commutes with $\mathcal{D}$, i.e., $K\mathcal{D} = \mathcal{D}K$:

$$\mathcal{K} := \{\text{diag}(K^1, \ldots, K^L) \mid$$
$$K^i \text{ invertible}, \ i = 1, \ldots, L\}. \qquad (13)$$

Theorem 7.1. System (11) *interconnected with the delay blocks* (12) *as in Fig. 3 with time-varying* $N_k^i \in [m_i, M_i] \cap \mathbb{N}$, $k \in \mathbb{N}$, *where* $m_i, M_i \in \mathbb{N}$ *and* $m_i \leq M_i$, $i = 1, \ldots, L$, *is globally asymptotically stable, if* A *Schur and*

$$\inf_{K \in \mathcal{K}} \|K\text{diag}(\sqrt{M_1 - m_1 + 1}I_{n_1}, \ldots,$$
$$\sqrt{M_L - m_L + 1}I_{n_L})H(z)K^{-1}\|_{\infty} \qquad (14)$$

is strictly smaller than 1.

The proof is omitted for space reasons. In a similar fashion, using Theorem 6.2, we obtain the following result.

Theorem 7.2. System (11) *interconnected with the delay blocks* (12) *as in Fig. 3 with time-varying* $N_k^i \in [m_i, M_i] \cap \mathbb{N}$, $k \in \mathbb{N}$, *where* $m_i, M_i \in \mathbb{N}$ *and* $m_i \leq M_i$, $i = 1, \ldots, L$, *is globally asymptotically stable, if* $A + BC$ *is Schur and*

$$\inf_{K \in \mathcal{K}} \|K \text{diag}(1 + \sqrt{M_1 - m_1 + 1}I_{n_1}, \ldots, \quad (15)$$

$$1 + \sqrt{M_L - m_L + 1}I_{n_L}) \cdot (I - H(z))^{-1} H(z) K^{-1}\|_\infty$$

is strictly smaller than 1.

Note that for $K = I_{n_q}$ and $m_i = m$ and $M_i = M$ for all $i = 1, \ldots, L$ the results of Theorem 7.1 and Theorem 7.2 reduce to Theorem 4.3 and Theorem 6.2, respectively. The scaling matrix K can be exploited to reduce conservatism compared to the case in which no scaling is used (i.e., $K = I_{n_q}$). For methods to compute the values (13) and (15) we refer the reader to Packard and Doyle (1993); Zhou et al. (1996); Skogestad and Postlethwaite (2005).

The frequency-domain characterizations in Theorem 7.1 and Theorem 7.2 also allow for systematic controller design. For instance, when $H(z) = P(z)C(z)$, where $C(z)$ is the transfer matrix of the controller, standard $\mathcal{H}_\infty$ and SSV-synthesis techniques from robust control can be used to design an appropriate controller $C(z)$ possibly also including other performance requirements related to (robust) tracking and disturbance suppression properties, see Packard and Doyle (1993); Zhou et al. (1996); Skogestad and Postlethwaite (2005).

8. CONCLUSIONS

In this paper alternative frequency-domain criteria for robust stability of discrete-time systems with time-varying delays are presented. These results are formulated in terms of $\mathcal{H}_\infty$ and structured singular value (SSV) conditions that can also directly be exploited for controller synthesis using the standard $\mathcal{H}_\infty$ and SSV design machinery as available in the robust control community. By a numerical example it is shown that the newly proposed stability conditions are less conservative than the existing frequency-domain stability criteria in certain situations. The presented modeling framework allowed for multiple sensor-to-controller and controller-to-actuator channels with different delay characteristics. In this manner various NCS architectures can be analyzed in an efficient manner using the computationally friendly frequency-domain conditions, thereby also including large-scale systems for which LMI-based stability conditions become prohibitively complex.

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