Frequency-Domain Tools for Stability Analysis of Hybrid Integrator-Gain Systems

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Abstract— Two new frequency-domain conditions for stability analysis of the feedback interconnection between a linear time-invariant (LTI) system and a sector-bounded hybrid integrator are presented. Compared to existing results, the conditions exploit more knowledge regarding the hybrid integrator's switching strategy, and guarantee the existence of a quadratic Lyapunov function that does not need to be positive within the full state-space. The conditions can be verified graphically on the basis of frequency response function data in a manner that is comparable to the classical Popov plot, making these tools valuable in practice.

I. INTRODUCTION

Hybrid integrator-gain systems, abbreviated with HIGS, are nonlinear, sector-bounded integrators that can be used for overcoming fundamental performance limitations of linear time-invariant (LTI) control systems [1]. Given the apparent integrator characteristics with a phase lag of only 38.15 degrees as observed from its describing function, HIGS is particularly appealing from a motion control perspective. These performance benefits, however, do not come without a cost. The nonlinear nature of the sector-bounded integrator renders many classical frequency-domain tools for stability and performance analysis inapplicable. As the current motion control practice highly exploits frequency-domain methods, lack of appropriate tooling may compromise applicability of nonlinear control strategies like HIGS in industry.

To accommodate for this situation, in [2] conditions for stability of HIGS-controlled systems that are graphically verifiable on the basis of (measured) frequency response function (FRF) data are proposed. A key step in the approach is to write the closed-loop system in a Lur'e form, being the feedback interconnection of an LTI system and HIGS. Sector-boundedness of HIGS is then exploited for formulating circle-criterion-like stability conditions, see also [12].

Although useful in practice, the conditions presented in [2] may render a rather conservative estimate on closed-loop stability. To some extent, this is caused by the fact that only sector-boundedness is taken into account in the analysis, while the interplay between the underlying integrator dynamics of HIGS and the LTI dynamics is being ignored. Hence, in the context of [2], HIGS is regarded as a generic sector-bounded nonlinearity. This is evidenced by the fact that the parameter associated with the integrator dynamics plays no decisive role in the stability analysis whatsoever.

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In this paper, two novel, distinct sets of frequency-domain conditions for stability analysis of the feedback interconnection of an LTI system and a dynamic nonlinearity such as HIGS are presented. The conditions exploit more information regarding the switching rules, as well as the interaction between the underlying dynamics of HIGS and the dynamics of the linear portion of the closed-loop system. The presented conditions relate to the results in [4], [5], [6], [7], [8], [9], [10], where frequency-domain conditions for classes of switched and hybrid systems under arbitrary- and state-dependent switching are given, which are all based on the existence of a common quadratic Lyapunov function. Especially, the first set of conditions presented in this paper can be seen as a special case of the results in [8], [9], but tailored to HIGS. The second set of frequency-domain conditions presented in this paper is truly novel in the sense that it makes explicit use of the S-procedure for guaranteeing the existence of a Lyapunov function that is guaranteed to be positive only in a subset of the state space, and for which its time-derivative is negative within this same subset. This relaxation increases the admissible class of Lyapunov functions for proving stability, and in that sense reduces conservatism in the analysis. The practical relevance of the conditions comes from the possibility to verify these graphically on the basis of (measured) FRF data. This is shown to be done in a manner that is comparable to the classical Popov plot [3].

Although mainly intended for HIGS, with some adjustments the ideas and results outlined in this paper may be applicable to a larger class of piecewise linear systems.

This paper is organized as follows. In Section II the control system setting and HIGS are discussed. In Section III, the main results of this paper are presented in the form of two theorems that set forth graphically verifiable frequency-domain conditions for stability. Application of the presented results is demonstrated on an example in Section IV. Section V states the main conclusions.

II. SYSTEM DESCRIPTION

The results in this paper are derived for the single-input single-output (SISO) control configuration as depicted in Fig. 1, which represents the feedback interconnection of a linear time-invariant (LTI) system G and the hybrid integrator-gain system $\mathcal{H}$. The latter is specified in more detail in Section II-A below.

The LTI system G is given in state-space format by

$$G : \begin{cases} \dot{x}_l & = A_l x_l + B_l u + B_w w, \\ y & = C_l x_l + D_l u + D_w w \end{cases} \tag{1}$$
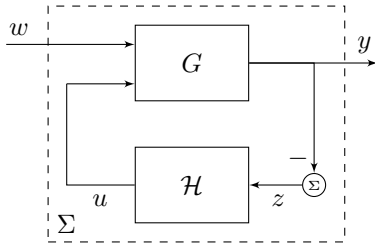
Fig. 1: Feedback interconnection of an LTI system G and HIGS $\mathcal{H}$.

with states $x_l(t) \in \mathbb{R}^n$, exogenous input $w(t) \in \mathbb{R}$, controlled output $y(t) = -z(t) \in \mathbb{R}$, and control input $u(t) \in \mathbb{R}$ at time $t \in \mathbb{R}_{\geq 0}$. It is assumed that (A_l, B_l, C_l, D_l) is minimal. Moreover,

$$G_{yu}(s) = C_l (sI - A_l)^{-1} B_l + D_l, \tag{2a}$$
$$G_{yw}(s) = C_l (sI - A_l)^{-1} B_w + D_w \tag{2b}$$

denote the transfers from u to y, and w to y, respectively. The following assumption on the transfer functions $G_{yu}(s)$ and $G_{yw}(s)$ in (2) is made.

Assumption 1. *The transfer functions $G_{yu}(s)$ and $G_{yw}(s)$ in* (2) *have a relative degree of at least two,* i.e., $D_l = D_w = 0$, *and $C_l B_l = C_l B_w = 0$.*

This assumption aids the proofs in this paper and is considered a mild assumption within a motion control context where plants are typically described by double integrators with additional structural dynamics, which naturally lead to a relative degree of two or higher.

A. Hybrid Integrator-Gain System (HIGS)

In this paper, the hybrid integrator-gain system is represented as the piecewise linear (PWL) system with discontinuous right-hand side, given by

$$\mathcal{H} : \begin{cases} \dot{x}_h &= \omega_h z, & \text{if } (z, u, \dot{z}) \in \mathcal{F}_1, \\ x_h &= k_h z, & \text{if } (z, u, \dot{z}) \in \mathcal{F}_2, \\ u &= x_h, \end{cases} \tag{3}$$

where $x_h(t) \in \mathbb{R}$ denotes the state of the integrator, $z(t) \in \mathbb{R}$ is the input. The input function z is assumed to be at least one-time differentiable, the time-derivative is denoted by $\dot{z}$, and $u(t) \in \mathbb{R}$ is the generated output. The parameter $\omega_h \in \mathbb{R}_{>0}$ is the integrator frequency, and $k_h \in \mathbb{R}_{>0}$ is the gain value. The active dynamics of HIGS are dictated by the sets $\mathcal{F}_1$ and $\mathcal{F}_2$, which are defined by

$$\mathcal{F}_1 := \left\{ (z, u, \dot{z}) \in \mathbb{R}^3 \mid k_h z u \geq u^2 \right\} \setminus \mathcal{F}_2, \tag{4a}$$
$$\mathcal{F}_2 := \left\{ (z, u, \dot{z}) \in \mathbb{R}^3 \mid u = k_h z \wedge \omega_h z^2 > k_h \dot{z} z \right\}. \tag{4b}$$

Note that the union of these sets, $\mathcal{F} := \mathcal{F}_1 \cup \mathcal{F}_2 = \left\{ (z, u, \dot{z}) \in \mathbb{R}^2 \mid k_h z u \geq u^2 \right\}$ defines the $[0, k_h]$-sector, and, therefore, the choice for $\mathcal{F}_1$ and $\mathcal{F}_2$ in (4) confines the input-output trajectory of HIGS to this sector. For a more detailed discussion on HIGS, the choice for the sets in (4), and different modelling setups, the reader is referred to *e.g.*, [2], [11], and the references therein.

B. Closed-loop system description

Due to the PWL nature of HIGS (3), and under Assumption 1, the closed-loop system in Fig. 1 naturally admits the PWL representation

$$\Sigma : \begin{cases} \dot{x} &= A_i x + B w, & \text{if } Ex \in \mathcal{F}_i, i \in \{1, 2\}, \\ y &= C x \end{cases} \tag{5}$$

with augmented state vector $x(t) = [x_l(t)^\top, x_h(t)]^\top \in \mathbb{R}^{n+1}$, exogenous input $w(t) \in \mathbb{R}$, and controlled output $y(t) \in \mathbb{R}$ at time $t \in \mathbb{R}_{\geq 0}$. The system matrices are given by

$$A_1 = \begin{bmatrix} A_l & B_l \\ -\omega_h C_l & 0 \end{bmatrix}, \text{ and } A_2 = \begin{bmatrix} A_l & B_l \\ -k_h C_l A_l & 0 \end{bmatrix}, \tag{6}$$

and furthermore $B = [B_l^\top, 0]^\top$, and $C = [C_l, 0]$. The matrix E extracts the states from x that determine mode switching of HIGS, that is, $Ex = [z, u, \dot{z}]^\top$, and is, therefore, given by

$$E^\top = \begin{bmatrix} -C_l^\top & 0 & -(C_l A_l)^\top \\ 0 & 1 & 0 \end{bmatrix}. \tag{7}$$

When the sub-dynamics governed by the matrix A_1 are active, the closed-loop system (5) is said to be in 'integrator-mode', whereas the dynamics governed by A_2 are referred to as 'gain-mode'. Note that A_2 results from explicit differentiation of the algebraic constraint $x_h = k_h z$ in (3). The closed-loop system (5) admits two alternative representations, namely,

$$\dot{x} = A_1 x - b v_1(x) + B w, \tag{8}$$

and

$$\dot{x} = A_2 x + b v_2(x) + B w, \tag{9}$$

where $v_1(x) = \varphi(Ex) c x$ and $v_2(x) = (1 - \varphi(Ex)) c x$ with

$$\varphi(Ex) = \begin{cases} 0, & \text{if } Ex \in \mathcal{F}_1, \\ 1, & \text{if } Ex \in \mathcal{F}_2, \end{cases} \tag{10}$$

$b = [0, \omega_h]^\top$ and $c = [C_y \left(\frac{k_h}{\omega_h} A - I \right), 0]$. Here, $\varphi(Ex)$ can be seen as an 'on/off' switch, which satisfies the sector constraint $\varphi(Ex) \in [0, 1]$, and $1 - \varphi(Ex) \in [0, 1]$. In fact, the representation in (8) or (9) allows for direct application of the circle-criterion, see *e.g.* [12, Theorem 8]. The interesting aspect is that the internal integrator/gain dynamics of HIGS (3) are encapsulated in the linear part of the closed-loop system, while state-dependent switching is captured by $\varphi(Ex)$. This viewpoint on the dynamics is instrumental in the upcoming analysis. Switched systems with state-dependent constraints are also considered in [8], [9].

C. Problem formulation

The main objective of this paper is to derive sufficient conditions for assessing input-to-state stability (ISS) of the closed-loop system in (5) using FRFs.

Definition 1 ([2])**.** *The closed-loop system in* (5) *is said to be input-to-state stable (ISS), if there exist a $\mathcal{KL}$-function α and a $\mathcal{K}$-function β such that for any initial condition*

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$x(0) = x_0 \in \mathbb{R}^{n+1}$ *and any bounded input signal* w, *all corresponding solutions to* (5) *satisfy for all* $t \in \mathbb{R}_{\geq 0}$

$$\|x(t)\| \leq \alpha(\|x(0)\|, t) + \beta\left(\sup_{0 \leq \tau \leq t} \|w(\tau)\|\right). \quad (11)$$

Due to the discontinuous nature of the PWL system (5), the existence of solutions is not immediate. In [11] it has been shown that for inputs w belonging to the class of piecewise Bohl functions [11, Definition 2], existence and forward completeness of solutions to (5) is formally guaranteed. In the remainder, it is assumed that locally absolutely continuous solutions to (5) exist for all $t \in \mathbb{R}_{\geq 0}$ and for all bounded inputs and initial states.

III. MAIN RESULTS

This section presents the main results of the paper in the form of two theorems consisting of different sufficient frequency-domain conditions for guaranteeing ISS.

Theorem 1. *Consider the closed-loop system* (5) *with Assumption 1 satisfied. Suppose that the following conditions hold:*

1) *The matrix* A_1 *given in* (6) *is Hurwitz;*
2) *There exist constants* $\tau_1 \geq 0$ *and* $\tau_2 \in \mathbb{R}$ *such that*

$$1 + \text{Re}\{W_1(j\omega)\} > 0 \quad \text{for all } \omega \in \mathbb{R}, \quad (12)$$

where

$$W_1(j\omega) := (c + \tau_1 c_z + \tau_2 h)(j\omega I - A_1)^{-1} b \quad (13)$$

with $b = [0, \omega_h]^\top$, $c = [C_y(\frac{k_h}{\omega_h}A - I), 0]$, $c_z = [-C_y, 0]$, *and* $h = [-k_h C_y, -1]$. *Then,* (5) *is ISS.*

Proof. The proof is given in Appendix A. $\square$

The conditions in Theorem 1 can be verified using (measured) FRFs in a Popov-like graphical manner. To see how, first note that the integrator-mode matrix A_1 represents the system matrix of the negative feedback interconnection of $G_{yu}(s)$ in (2a) and HIGS replaced by the LTI integrator ω_h/s, see also Fig. 1. Stability of this linear feedback interconnection, and thus the Hurwitz property of A_1, can be verified by applying the Nyquist stability criterion [13, Section 6.3] to the open-loop characteristic $L(j\omega) = \frac{\omega_h}{j\omega}G_{yu}(j\omega)$.

Next, observe that $W_1(s)$ in (13) represents the transfer function from an input v that enters the input channel in the LTI integrator, to the output $q = z - \frac{k_h}{\omega_h}\dot{z} + \tau_1 z + \tau_2(k_h z - u)$. On the basis of Fig. 1, with $w = 0$, $\mathcal{H} = \mathcal{H}_1(s) = \omega_h/s$, and $z = -y$ one finds

$$W_1(s) = \left(1 + \tau_1 + \tau_2 k_h - \frac{k_h}{\omega_h}s\right)G_{zv}(s) - \tau_2 G_{uv}(s),$$

with

$$G_{zv}(s) = c_z(sI - A_1)^{-1}b = -\frac{\mathcal{H}_1(s)G_{yu}(s)}{1 + \mathcal{H}_1(s)G_{yu}(s)}, \quad (14a)$$

$$G_{uv}(s) = \mathcal{H}_1(s)(G_{zv}(s) + 1) = \frac{\mathcal{H}_1(s)}{1 + \mathcal{H}_1(s)G_{yu}(s)}, \quad (14b)$$

and where $G_{yu}(s)$ is given in (2a). Verifying (12) in condition 2) amounts to verifying if there exist $\tau_1 \geq 0$ and $\tau_2 \in \mathbb{R}$ such that

$$1 + \text{Re}\left\{\left(1 - k_h\frac{j\omega}{\omega_h}\right)G_{zv}(j\omega) + \tau_1 G_{zv}(j\omega) \right.$$
$$\left. + \tau_2\left(k_h - \frac{\omega_h}{j\omega}\right)G_{zv}(j\omega) - \tau_2\frac{\omega_h}{j\omega}\right\} > 0 \quad (15)$$

holds. By noting that $G_{zv}(j\omega) = \text{Re}\{G_{zv}(j\omega)\} + j\text{Im}\{G_{zv}(j\omega)\}$ one finds that (15) is equivalent to

$$1 + X_1(j\omega) - \tau_1 Y_1(j\omega) - \tau_2 Z_1(j\omega) > 0, \quad (16)$$

where

$$X_1(j\omega) := \text{Re}\{G_{vz}(j\omega)\} + k_h\frac{\omega}{\omega_h}\text{Im}\{G_{vz}(j\omega)\}, \quad (17a)$$

$$Y_1(j\omega) := -\text{Re}\{G_{vz}(j\omega)\}, \quad (17b)$$

$$Z_1(j\omega) := \frac{\omega_h}{\omega}\text{Im}\{G_{vz}(j\omega)\} - k_h\text{Re}\{G_{vz}(j\omega)\}. \quad (17c)$$

One can verify that condition (16) implies that in the (x, y, z)-space, the curve given by (17) should lie to the right of the two-dimensional plane defined by $1 + x - \tau_1 y - \tau_2 z = 0$, which can be verified graphically. Note, however, that finding τ_1, τ_2 for ensuring this property, might be a cumbersome task. On the other hand, by fixing either τ_1 or τ_2 a priori, the graphical test becomes easier to verify at the cost of possibly introducing conservatism. For example, when fixing τ_1, the two-dimensional curve given by $(\tilde{X}_1, Z_1)$ with $\tilde{X}_1 := X_1 - \tau_1 Y_1$ should lie to the right of the line $1 + x - \tau_2 z = 0$, *i.e.*, a line that passes through $(-1, 0)$ with a slope of $1/\tau_2$. A similar test follows when fixing τ_2. Due to the resemblance with the traditional Popov plot, Theorem 1 (after fixing one of the two parameters τ_1, τ_2) thus provides a graphical Popov-like stability test.

Theorem 1 guarantees the existence of a common quadratic Lyapunov function (CQLF) for the pair of system matrices A_1 and $A_1 - b(c + \tau_1 c_z + \tau_2 h) = A_2 - b(\tau_1 c_z + \tau_2 h)$, see also [9, Comment 1]. Clearly, this shows that the gain-mode matrix A_2 itself does not need to be Hurwitz. For $\tau_1 = \tau_2 = 0$ the conditions reduce to the circle-criterion which guarantees the existence of a (CQLF) for the pair (A_1, A_2), and thus verifies stability for a system that switches arbitrarily between integrator-mode and gain-mode. Theorem 1 exploits the interaction of the LTI dynamics in G (1) and HIGS' integrator dynamics in (3) in an explicit manner. Compared to the existing frequency-domain conditions for HIGS in [2, Theorem 1], the system matrix A in (1) does not have to be Hurwitz, which typically provides an advantage when using dynamic nonlinearities such as HIGS in open-loop configurations. A possible drawback, however, is the need for A_1 to be Hurwitz, whereas it is known that an unstable A_1 may still yield a stable closed-loop system, and may even largely contribute to improved closed-loop performance. To address this issue, the next result is developed.

Theorem 2. *Consider the closed-loop system* (5) *with Assumption 1 satisfied. Suppose that the following conditions hold:*

1) *The matrix $A - k_h B_u C_y$ with (A, B_u, C_y) given in* (1) *is Hurwitz;*

2) *There exists a constant $\tau \geq 0$ such that*

$$1 + \text{Re}\{W_2(j\omega)\} > 0 \text{ for all } \omega \in \mathbb{R}, \quad (18)$$

where

$$W_2(j\omega) = (\tau\omega_h(c_z + c_u) + c_s + c_z - 2c_u)(j\omega I - \hat{A}_2)^{-1} b \quad (19)$$

with $\hat{A}_2 := A_2 + \frac{2}{k_h} bh$, $b = [0, \omega_h]^\top$, $h = [-k_h C_y, -1]$, $c_z = [-C_y, 0]$, $c_u = [0, \frac{1}{k_h}]$, *and* $c_s = [-\frac{k_h}{\omega_h} C_y A, 0]$. *Then,* (5) *is ISS.*

Proof. The proof is given in Appendix B. □

Similar as before, the conditions in Theorem 2 can be verified in a graphical Popov-like manner. First observe that the matrix $A - k_h B_u C_y$ represents the negative feedback interconnection of G_{yu} given in (2a) with a gain k_h. Hence, for verifying the Hurwitz property of this matrix, one can apply the Nyquist stability criterion to the open-loop characteristic $L(j\omega) = k_h G_{yu}(j\omega)$.

Next, observe that $\hat{A}_2$ can be seen as the system matrix of the negative feedback interconnection of the system $\frac{k_h}{\omega_h}(s + 2\frac{\omega_h}{k_h})G_{yu}(s)$ with HIGS being replaced by a low-pass filter, *i.e.*, $\mathcal{H} = \mathcal{H}_2(s) = \omega_h/(s + 2\omega_h/k_h)$. As such, it follows that $W_2(s)$ in (19) represents the transfer function from an input v entering the input channel in $\mathcal{H}_2(s)$ to the output $q = \tilde{\tau}(z + u/k_h) + k_h\dot{z}/\omega_h + z - 2u/k_h$, with $\tilde{\tau} = \tau\omega_h$. After some manipulation one finds

$$W_2(s) = \left(\tilde{\tau} + 1 + \frac{k_h}{\omega_h}s\right) G_{zv}(s) - \left(\frac{2 - \tilde{\tau}}{k_h}\right) G_{uv}(s),$$

with

$$G_{zv}(s) = c_z(sI - \hat{A}_2)^{-1} b = -\frac{\mathcal{H}_2(s)G_{yu}(s)}{1 + k_h G_{yu}(s)}, \quad (20a)$$

$$G_{uv}(s) = k_h G_{zv}(s) + \mathcal{H}_2(s) = \frac{\mathcal{H}_2(s)}{1 + k_h G_{yu}(s)}, \quad (20b)$$

and where $G_{yu}(s)$ is given in (2a). Verifying condition (18) amounts to finding $\tau \geq 0$ such that

$$1 + X_2(j\omega) - \tau\omega_h Y_2(j\omega) > 0, \quad (21)$$

where

$$X_2(j\omega) := -\text{Re}\{G_{zv}(j\omega)\} - k_h \frac{\omega}{\omega_h} \text{Im}\{G_{zv}(j\omega)\} - \frac{(2\omega_h/k_h)^2}{\omega^2 + (2\omega_h/k_h)^2}, \quad (22a)$$

$$Y_2(j\omega) := -2\text{Re}\{G_{zv}(j\omega)\} - \frac{2 \cdot (\omega_h/k_h)^2}{\omega^2 + (2\omega_h/k_h)^2}. \quad (22b)$$

Then, in the (x, y)-plane, the curve defined by (22) should lie to the right of a line that passes through the point $(-1, 0)$ with a slope of $1/(\tau\omega_h)$.

The key differences between Theorem 1 and Theorem 2 regarding HIGS are as follows. First, Theorem 1 assumes only the integrator-mode of HIGS to be stable, whereas Theorem 2 assumes only the gain-mode of HIGS to be

stable. Hence, both results are applicable to different cases. Second, the frequency-domain conditions in Theorem 2 guarantee the existence of a Lyapunov function that satisfies $V(x) > 0$ for all $x \in \{x \in \mathbb{R}^{n+1} \mid Ex \in \mathcal{F}\} \setminus \{0\}$, *i.e.*, one that is only guaranteed to be positive definite in the sector, see Appendix B for details. On the other hand, Theorem 1 and many other existing frequency-domain conditions for switched systems in general, and HIGS in particular, typically guarantee the existence of a common Lyapunov function satisfying $V(x) > 0$ for all $x \in \mathbb{R}^m \setminus \{0\}$, *i.e.*, positive definite within the full state-space, which sometimes is too restrictive.

IV. EXAMPLE

In this section, applicability of the presented tools is demonstrated on a motion control example. Conform Fig. 1, the linear transfer function from u to y, see (2a), is given by $G_{yu}(j\omega) = C(j\omega)P(j\omega)$, where $P(j\omega)$ represents the experimental motor-load motion system depicted in Fig. 2, which consists of two rotating inertias connected by a thin, flexible shaft, and $C(j\omega)$ is the linear part of the controller. Non-collocated actuation is considered, *i.e.*, measurements by the encoder at the load side (right side) are separated from actuation by the motor side (left side). The measured FRF data is provided in Fig. 3.
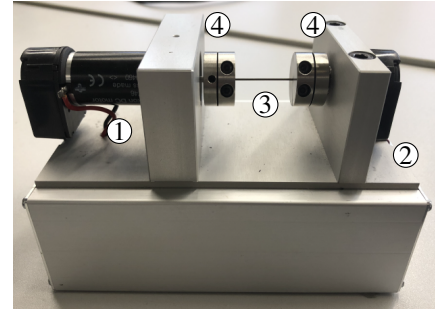

Fig. 2: Motor-load motion system with ①: actuator (motor side), ②: sensor (load side), ③: flexible shaft, and ④: rotating inertias.

The linear part of the controller is given by $C(s) = C_{pid}(s)C_n(s)$ with

$$C_{pid}(s) = k_p\left(1 + \frac{\omega_i}{s} + \frac{s}{\omega_d}\right)\frac{s + \omega_c}{\omega_c}\frac{\omega_{lp}^2}{s^2 + 2\beta\omega_{lp}s + \omega_{lp}^2},$$

$$C_n(s) = \left(\frac{\omega_p}{\omega_z}\right)^2 \cdot \frac{s^2 + 2\beta_z\omega_z s + \omega_z^2}{s^2 + 2\beta_p\omega_p s + \omega_p^2}.$$

Note that $C(s)$ in series with HIGS represents a PID-type of controller with a HIGS-based low-pass filter that aims at improved phase properties, see also the design rationale presented in [1]. The following parameter values are used: $k_p = 1.1$ N/m, $\omega_i = 3 \cdot 2\pi$ rad/s, $\omega_d = 6 \cdot 2\pi$ rad/s, $\omega_{lp} = 38 \cdot 2\pi$ rad/s, $\beta = 0.9$, $\omega_z = 63 \cdot 2\pi$, $\beta_z = 0.01$, $\omega_p = 90 \cdot 2\pi$, $\beta_p = 0.1$, and $\omega_c = \omega_h|1 + 4j/\pi|/k_h$ with $\omega_h = 40 \cdot 2\pi$ rad/s, and $k_h = 1$. The open-loop characteristics of the integrator mode, being $L_i(j\omega) := C(j\omega)P(j\omega)\omega_h/j\omega$, and the gain-mode, $L_k(j\omega) := k_h C(j\omega)P(j\omega)$ are shown in Fig. 4.
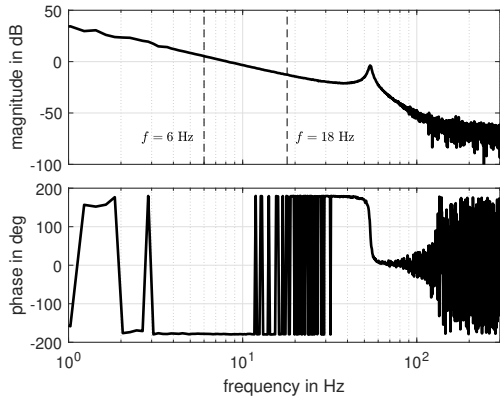
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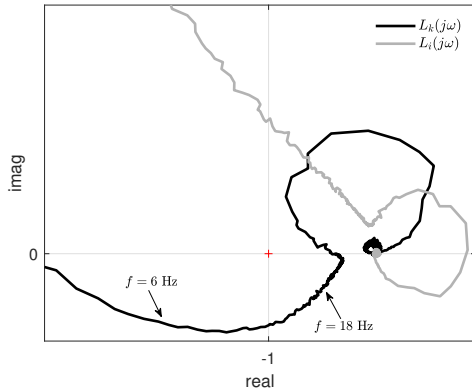Fig. 3: Measured frequency response function of $P(j\omega)$.

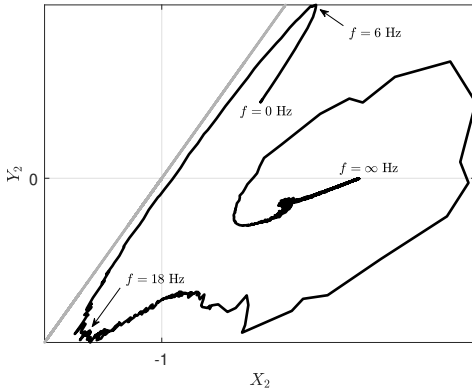Fig. 4: Open-loop characteristics of the integrator-and gain-mode.

Fig. 5: Popov-like plot of the (X_2, Y_2)-curve.

By the Nyquist stability criterion, it follows that the integrator-mode dynamics are unstable, whereas the gain-mode dynamics are stable. Hence, Theorem 1 cannot be applied, and thus one need to use Theorem 2 for asserting closed-loop stability. For $\tau = \frac{0.55}{\omega_h}$ the conditions of Theorem 2 are satisfied. The corresponding (X_2, Y_2)-plot (22) is given in Fig. 5. Observe that in the critical frequency range $f \in [6, 18]$ Hz, there is some margin before violating the conditions, which implies a certain amount of robustness. This range corresponds to the rigid-body mode of the plant, which is typically measured with high-accuracy. Using Theorem 2, stability is verified for all $\omega_h \geq 28 \cdot 2\pi$ rad/s.

V. CONCLUSIONS

In this paper, two novel frequency-domain conditions for verifying stability of hybrid integrator-gain systems are presented. The conditions explicitly take into account the sector properties in integrator- and/or gain-mode, and the interaction between HIGS' dynamics and the dynamics of the interconnected LTI system. In fact, the existence of quadratic Lyapunov functions is guaranteed that may satisfy the corresponding demands only in those subregions of the state-space in which trajectories can evolve. The conditions can be verified in a manner that is comparable with the classical Popov plot, making them valuable tools in practice.

APPENDIX

A. Proof of Theorem 1

First observe that, under the conditions of the theorem and Assumption 1, one has $1 + \mathrm{Re}\left\{\lim_{\omega \to \infty} W_1(j\omega)\right\} > 0$. Indeed, due to the assumption on the relative degree of G_{yu} one has $\lim_{|\omega| \to \infty} G_{yu}(j\omega) = 0$. Since also $\lim_{|\omega| \to \infty} \omega_h/j\omega = 0$, one can eventually conclude that $\lim_{|\omega| \to \infty} W_1(j\omega) = 0$. Together with (12), this implies the transfer function $1 + W_1(s)$ to be strictly positive real (SPR) [3, Definition 6.4]. By virtue of the Kalman-Yakubovich-Popov (KYP) lemma [3, Lemma 6.3], this implies the existence of a positive definite matrix $P = P^\top$, matrix L, and a constant $\epsilon > 0$ such that

$$A_1^\top P + PA_1 = -L^\top L - \epsilon P \tag{24a}$$
$$Pb = (c + \tau_1 c_z + \tau_2 h)^\top - \sqrt{2}L^\top. \tag{24b}$$

Consider the quadratic function $V(x) = x^\top Px$ that will be shown to be an ISS-Lyapunov function. Note that $V(x) > 0$ for all $x \neq 0$. The time-derivative of V along the solutions of (5) in integrator-mode ($i = 1$) satisfies

$$\begin{aligned}
\dot{V} &= x^\top\left(A_1^\top P + PA_1\right)x + 2x^\top PBw \\
&\stackrel{(24)}{=} -x^\top L^\top Lx - \epsilon V + 2x^\top PBw \\
&\leq -\epsilon V + 2\|PB\|\|x\|\|w\| \leq -\mu V + \gamma\|w\|^2,
\end{aligned} \tag{25}$$

where for the second inequality use is made of Young's inequality, and with $\mu = \epsilon - \delta/\lambda_{\min}(P)$, and $\gamma = 2\|PB\|^2/\delta$. Choosing $0 < \delta < \epsilon\lambda_{\min}(P)$ guarantees $\mu, \gamma > 0$.

Next, consider the time-derivative of V along the trajectories of (5) in gain-mode ($i = 2$), which gives

$$\begin{aligned}
\dot{V} &= x^\top(A_2^\top P + PA_2)x + 2x^\top PBw \\
&= x^\top(A_1^\top P + PA_1)x - 2x^\top Pbcx + 2x^\top PBw,
\end{aligned}$$

where it is used that $A_2 = A_1 - bc$. Substituting (24) yields

$$\begin{aligned}
\dot{V} &= -x^\top L^\top Lx - \epsilon V + 2x^\top PBw \\
&\quad - 2(x^\top((c + \tau_1 c_z + \tau_2 h)^\top - L^\top\sqrt{2})cx) \\
&= -x^\top(L - \sqrt{2}c)^\top(L - \sqrt{2}c)x - \epsilon V \\
&\quad - 2\tau_1 x^\top c^\top c_z x - 2\tau_2 x^\top c^\top hx + 2x^\top PBw \\
&\leq -\epsilon V + 2x^\top PBw - 2\tau_1 z\left(z - \frac{k_h}{\omega_h}\dot{z}\right) - 2\tau_2 x^\top h^\top cx \\
&\leq -\mu V + \gamma\|w\|^2,
\end{aligned}$$

with $\mu = \epsilon - \delta/\lambda_{\min}(P)$ and $\gamma = 2\|PB\|^2/\delta$, and where use is made of the fact that in gain-mode $hx = 0$ and $k_h \dot{z} z - \omega_h z^2 < 0$, $\tau_1 \geq 0$ and $\tau_2 \in \mathbb{R}$. The time-derivative of V in each mode satisfies (almost everywhere) a common upper-bound given by

$$\dot{V} \leq -\mu\|x\|^2 + \gamma\|w\|^2, \qquad (26)$$

and, therefore, the function $V(x)$ is a suitable ISS-Lyapunov function. This completes the proof.

B. Proof of Theorem 2

The proof is based on showing that the conditions in the theorem imply the existence of an ISS-Lyapunov function of the form

$$V(x) = x^\top P x + \tau \left(k_h z^2 + 2zx_h - \frac{3}{k_h} x_h^2 \right) \qquad (27)$$
$$= x^\top (P + \tau Q) x,$$

with $Q := k_h c_z^\top c_z + k_h \left(c_z^\top c_u + c_u^\top c_z \right) - 3k_h c_u^\top c_u$.

First note that under the condition that $A - k_h B_u C_y$ is Hurwitz, the matrix $\hat{A}_2$ is also Hurwitz. To see why, consider the transformation $\tilde{x} = Tx$ with $T = \begin{bmatrix} I & 0 \\ k_h C_y & 1 \end{bmatrix}$, such that

$$T\hat{A}_2 T^{-1} = \begin{bmatrix} A - k_h B_u C_y & B_u \\ 0 & -2\frac{\omega_h}{k_h} \end{bmatrix}.$$

Due to the upper-triangular structure, the eigenvalues of $T\hat{A}_2 T^{-1}$, and thus also of $\hat{A}_2$ are given by the eigenvalues of $A - k_h B_u C_y$ and $-2\omega_h/k_h$. As such, $\hat{A}_2$ is Hurwitz.

Similar as before, under the conditions of the theorem and Assumption 1, one finds that $1 + \text{Re}\left\{ \lim_{\omega \to \infty} W_2(j\omega) \right\} > 0$. Together with (18) this implies the transfer function $1 + W_2(s)$ to be SPR. By virtue of the KYP-lemma as in the above proof, there exist a positive definite matrix $P = P^\top$, matrix L, and a real constant $\epsilon > 0$ such that

$$\hat{A}_2^\top P + P\hat{A}_2 = -L^\top L - \epsilon P \qquad (28a)$$

$$Pb = \tilde{\tau}(c_z + c_u)^\top + \left(F + \frac{1}{k_h} h \right)^\top - \sqrt{2} L^\top, \quad (28b)$$

where $\tilde{\tau} = \tau\omega_h$ and $F := c_s - c_u$.

Consider the candidate ISS-Lyapunov function (27). Since P is positive definite and $\tau \geq 0$, and due to the sector condition $k_h z^2 \geq zx_h \geq x_h^2/k_h$, it follows from the S-procedure that V is positive definite for all $x \in \mathbb{R}^m \setminus \{0\}$ with $Ex \in \mathcal{F}$. The time-derivative of V in integrator-mode $(i = 1)$ satisfies

$$\dot{V} = x^\top (A_1^\top (P + \tau Q) + (P + \tau Q)A_1)x + 2x^\top PBw$$
$$= x^\top (\hat{A}_2^\top P + P\hat{A}_2)x - 2x^\top \left(Pb \left(F + \frac{1}{k_h} h \right) \right)x$$
$$+ 2\tau\omega_h x^\top \left((c_z + c_u)^\top F + \frac{1}{k_h^2} h^\top h \right)x + 2x^\top PBw,$$

where the identities $A_1 = \hat{A}_2 - b(F + h/k_h)$, and

$$A_1^\top Q + QA_1 = \omega_h \text{He}\left((c_z + c_u)^\top F + \frac{1}{k_h^2} h^\top h \right),$$

with $\text{He}(X) = X^\top + X$ are used. Substituting the equalities (28) yields

$$\dot{V} = -x^\top L^\top Lx - \epsilon x^\top Px + 2x^\top PBw$$
$$- 2x^\top \left(\left(F + \frac{1}{k_h} h \right)^\top - L^\top \sqrt{2} \right) \left(F + \frac{1}{k_h} h \right)x$$
$$- 4\tau\omega_h x^\top c_u^\top (c_z - c_u)x$$
$$\leq -\epsilon x^\top Px + 2x^\top PBw$$
$$- x^\top \left(L - \sqrt{2} \left(F + \frac{h}{k_h} \right) \right)^\top \left(L - \sqrt{2} \left(F + \frac{h}{k_h} \right) \right)x$$
$$\leq -\mu V + \gamma\|w\|^2,$$

where use is made of the sector-condition $k_h z^2 - zx_h \geq 0$, and $\mu, \gamma > 0$ are obtained in a standard manner.

The time-derivative of V in gain-mode $(i = 2)$ satisfies

$$\dot{V} = x^\top (\hat{A}_2^\top P + P\hat{A}_2)x - \frac{2}{k_h} x^\top Pbhx + 2x^\top PBw.$$

Substituting the equalities in (28), using bounds on V as before, and using the fact that in gain-mode $hx = 0$, leads to the upper-bound in gain-mode as

$$\dot{V} \leq -\mu V + \gamma\|w\|^2, \qquad (29)$$

with $\mu, \gamma > 0$ as before. As such, (29) is a uniform upper-bound on $\dot{V}$ almost everywhere and thus V classifies as a suitable ISS-Lyapunov function, which completes the proof.

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