Stability of Networked Control Systems With Uncertain Time-Varying Delays

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Abstract—In this technical note, a new approach for the stability analysis and controller synthesis of networked control systems (NCSs) with uncertain, time-varying, network delays is presented. Based on the Jordan form of the continuous-time plant, a discrete-time representation of the NCS is derived. Using this model for delays that can be both smaller and larger than the sampling interval, sufficient LMI conditions for stability and feedback stabilization are proposed. The results are illustrated by a typical motion control example.

Index Terms—Linear matrix inequalities, networked control systems, sampled-data control, stability analysis, time-varying delay.

I. INTRODUCTION

Networked control systems (NCSs) are control systems in which the control loop is closed over a real-time network. Their advantages are a flexible architecture and a reduction of installation and maintenance costs, [1], [2]. The main disadvantages of NCSs are the network effects that influence the performance and stability of the control loop, such as time-delays and packet dropouts. Despite these disadvantages, NCSs are applied in a broad range of systems, such as mobile sensor networks, remote surgery, automated highway systems and unmanned aerial vehicles, see e.g. [1]–[3]. In the current technical note, we will focus on the modeling and stability analysis of a NCS with time-varying delays. The need for methods for the stability analysis of NCSs is motivated by the fact that a control system can be destabilized as a consequence of constant time-delays. However, the situation can become more interesting as in [4] and [5] it is shown that a system, that is stable for the best- and worst-case constant delays (and all constant values in between), can become unstable if the delay is time-varying within these bounds.

In the literature, different modeling and analysis approaches for NCSs with network delays can be distinguished. The majority of available literature [4]–[13] uses discretizations of the continuous-time plant. However, also models in the form of delay impulsive differential equations are proposed in [14] and continuous-time NCS models [15] are available as well.

The most common discrete-time NCS model is explained in e.g. [6] and [7]. Herein, a NCS configuration with a time-driven sensor and an event-driven controller and actuator is considered, where the time-delay is upperbounded by the sampling interval. For this model, different stability and related controller synthesis approaches are available in literature, see e.g. [5], [8], [9]. All three papers, consider a Lyapunov-based approach, but distinguish themselves by the method used

to deal with the time-variation in the delays. In [8] Taylor series are used, in [9] the maximum singular value of the continuous-time system in combination with the upper- and lowerbounds on the delay, while in [5] the notion of interval matrices is used. In [10], [11], the discrete-time model of [6], [7] is extended for time-varying delays larger than the sampling interval, however, the variation of the delay is limited by the sampling interval. This limitation is removed in [12], where a discrete-time model is proposed that describes the effect of multiple control inputs during one sampling interval. However, message rejection, being the effect that more recent control data becomes available before the older data is implemented and therefore the older data is neglected, is not considered in [12]. Another discrete-time analysis approach that considers delays with variations larger than the sampling interval is described in [13], without an explicit definition of the NCS model. Based on robust stabilization techniques and a Lyapunov-Kraskovskii approach, sufficient conditions for the stability analysis and controller synthesis are proposed. A third discrete-time approach for arbitrary large delays is proposed in [16], although this is limited to NCSs with discrete-time plants and assumes that message rejection does not occur.

A stability approach based on the small gain theorem is proposed in [17]. Here, a discretization of the nondelayed system is used, which allows for stability analysis of both small and large delays. A disadvantage of this approach is the fact that it is limited to systems with a strictly proper and stable continuous-time plant.

A continuous-time NCS modeling approach is given in [15] and [18]. In [15], a maximum allowable transmission interval is derived, which gives the maximum amount of time between two consecutive sensor messages for which stability can be guaranteed. In [18], a Lyapunov-based controller design is proposed for NCSs with time-varying delays and packet dropouts. Input-to-state (and input-output) stability properties of (nonlinear) NCS models, described in terms of impulsive differential equations, have been studied in [19] for NCSs with multiple packet communications, time-varying sampling intervals and different network protocols. An impulsive *delay*-differential approach is proposed in [14] for NCSs with variable sampling intervals, time-varying delays and packet dropouts. A main advantage of this modeling approach is the possibility to incorporate time-delays larger than the sampling interval without increasing model complexity, as is the case in the discrete-time modeling approach.

In this technical note, we propose a Lyapunov-based stability criterion in terms of linear matrix inequalities (LMIs) for discrete-time NCS models with bounded time-varying delays that can be both larger and smaller than the sampling interval. Note that the discrete-time NCS model and the delay-impulsive model of [14] represent alternative models. In this technical note, we rewrite the discrete-time NCS model using a (real) Jordan form of the continuous-time system matrices. Using this approach, the time-variation in the delays can be represented as a combination of uncertainty functions. For analysis purposes, based on these uncertainty functions, a convex overapproximation of the discrete-time model is used that explicitly contains the bounds of the time-varying delays. Compared to [5], [8], [9], the large delay case is included and alternative stability results are presented. We propose an extension to the model of [12], such that message rejection is included. Moreover, we decrease the number of uncertain parameters in the stability analysis, which is beneficial for the reduction of conservatism. Compared to the work in [13], we give an explicit definition of the uncertain functions instead of an implicit one. Besides stability conditions, we also provide sufficient conditions in the form of LMIs for the synthesis of stabilizing controllers.

This technical note is organized as follows. In Section II, we introduce the NCS model for time-varying delays smaller and larger than
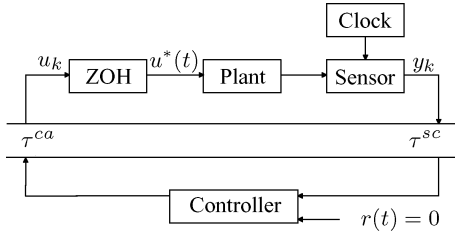
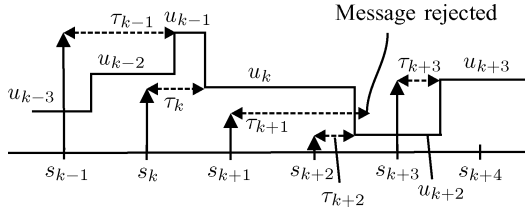Fig. 1.   Schematic overview of the networked control system.

Fig. 2.   Time-delays smaller and larger than the sampling interval, with $\tau_k \in [0, 2h]$.

the sampling interval. Section III gives the stability analysis and controller synthesis results. In Section IV, simulation results are presented that illustrate the effectiveness of the proposed approach for a motion control case study. Finally, Section V gives the conclusions.

Notation: By $diag$ we denote a block diagonal matrix with its input arguments on the diagonal. For a square matrix $S \in \mathbb{R}^{s \times s}$, we use $\dim(S) = s$, which we call the dimension of S. The superscript ^{T} denotes the transpose of a matrix. $0_{m,n}$ is the zero matrix of size $m \times n$ and I_m is the $m \times m$ identity matrix.

II. NCS MODEL

A NCS with a continuous-time plant and a discrete-time controller is schematically depicted in Fig. 1. In the model the computation time and the network-induced delays, i.e. the sensor-to-controller delay (τ^{sc}) and the controller-to-actuator delay (τ^{ca}), are represented as a single delay τ_k. This is possible, because the sensor is assumed to act in a time-driven fashion (i.e. sampling at the times $s_k := kh$, $k \in \mathbb{N}$, with $h > 0$, the constant sampling interval), the controller and actuator in an event-driven fashion (i.e. responding instantaneously to newly arrived data) and the controller is static and time-invariant—see [6] and [7].

Before the model of a NCS for delays smaller and larger than the sampling interval is presented, an example of the timing behavior in NCSs is given. In Fig. 2, the time-delays are allowed to take values in $[0, 2h]$. This figure shows that the number of active control inputs in one sampling interval is variable and depends on the previous and current time-delays. Moreover, message rejection, i.e., the effect that more recent control data becomes available before the older data is

implemented and therefore the older data is neglected, can occur. We define $k^*(t) := \max\{k \in \mathbb{N} | s_k + \tau_k \le t\}$ as the index of the most recent control input that is available at time t. Using this definition, the continuous-time NCS model becomes

$$\dot{x}(t) = Ax(t) + Bu^*(t),$$
$$u^*(t) = u_{k^*(t)} \tag{1}$$

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, the system matrices, $u^*(t) \in \mathbb{R}^m$ the continuous-time control input and $x(t) \in \mathbb{R}^n$ the state at time $t \in \mathbb{R}$ and $u_k \in \mathbb{R}^m$ the discrete-time control input corresponding to the measurement data at sampling instant s_k. Note that $u^*(t)$ is a piecewise constant signal. Due to the definition of k^*, the possibility of message rejection is explicitly included, while it is only implicitly included in [14], by assuming that the values of k are always sequential.

Before we present the discrete-time NCS model, we consider the exact discretization of (1) for the example of Fig. 2, with $\tau_k \in [0, 2h]$: see (2), as shown at the bottom of the page, with $\tau^*_{k-1} = \tau_{k-1} - h$. The last subsystem (2) describes message rejection, which results in some control data not affecting the evolution of the state (u_{k-1} in this case). The second subsystem corresponds to the large delay model that is presented in e.g. [7], [10], [11]. The first subsystem corresponds to the case with delays smaller than the sampling interval, as used in, e.g., [6] and [7]. For arbitrary, though bounded, time-varying delays, a NCS model that considers these different subsystems, except message rejection, is described in [12]. We will present a model description, which is based on an exact discretization of (1), to develop a NCS model, *including message rejection*, which is an essential feature for NCS modeling, as it ensures the implementation of the most recent data on the system.

To derive the discrete-time NCS model for large delays [incorporating all possibilities as in (2)], in Lemma 1, the general description of the continuous-time NCS as (1) is reformulated to make explicit which control inputs can be active in the sampling interval $[s_k, s_{k+1})$.

Lemma 1: Consider the continuous-time NCS as defined in (1). Define $\underline{d} := \lfloor \tau_{min}/h \rfloor$ and $\overline{d} := \lceil \tau_{max}/h \rceil$. Then, the control action $u^*(t)$ in the sampling interval $[s_k, s_{k+1})$ is given by

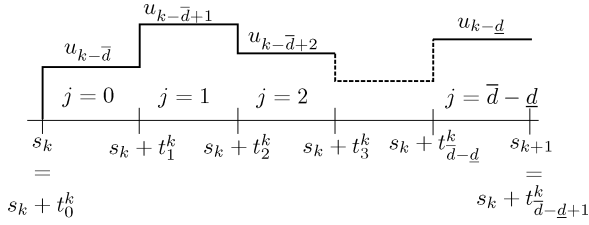$$u^*(t) = u_{k+j-\overline{d}} \text{ for } t \in \left[s_k + t^k_j, s_k + t^k_{j+1} \right), \tag{3}$$

with t^k_j defined as

$$t^k_j = \min \Big\{ \max\Big\{ 0, \tau_{k+j-\overline{d}} + (j - \overline{d})h \Big\},$$
$$\max\Big\{ 0, \tau_{k+j-\overline{d}+1} + (j - \overline{d} + 1)h \Big\}, \dots,$$
$$\max\Big\{ 0, \tau_{k-\underline{d}} - \underline{d}h \Big\}, h \Big\} \tag{4}$$

with $t^k_j \le t^k_{j+1}$ and $j \in \{0, 1, \dots, \overline{d} - \underline{d}\}$. Moreover, $t^k_0 := 0$ and $t^k_{\overline{d}-\underline{d}+1} := h$.

Proof: The proof is given in the Appendix.   ∎

$$x_{k+1} = \begin{cases} e^{Ah}x_k + \int_0^{h-\tau_k} e^{As}dsBu_k + \int_{h-\tau_k}^h e^{As}dsBu_{k-1}, & \text{if } \tau_k \le h \text{ and } \tau_{k-1} \le h \\ e^{Ah}x_k + \int_0^{h-\tau^*_{k-1}} e^{As}dsBu_{k-1} + \int_{h-\tau^*_{k-1}}^h e^{As}dsBu_{k-2}, & \text{if } h \le \tau_k \le 2h \text{ and } h \le \tau_{k-1} \le 2h \\ e^{Ah}x_k + \int_0^h e^{As}dsBu_{k-1}, & \text{if } h \le \tau_k \le 2h \text{ and } \tau_{k-1} \le h \\ e^{Ah}x_k + \int_0^{h-\tau_k} e^{As}dsBu_k \\ \qquad + \int_{h-\tau_k}^{h-\tau^*_{k-1}} e^{As}dsBu_{k-1} + \int_{h-\tau^*_{k-1}}^h e^{As}dsBu_{k-2}, & \text{if } \tau_k \le h \text{ and } h \le \tau_{k-1} \le 2h \text{ and } \tau_{k-1} - h < \tau_k \\ e^{Ah}x_k + \int_0^{h-\tau_k} e^{As}dsBu_k + \int_{h-\tau_k}^h e^{As}dsBu_{k-2}, & \text{if } \tau_{k-1} - h \ge \tau_k \end{cases} \tag{2}$$

Fig. 3. Example of the use of t_j^k.

Based on this lemma, we can define the discrete-time NCS model using $x_k = x(s_k)$, $k \in \mathbb{N}$ for large delays as

$$x_{k+1} = e^{Ah} x_k + \sum_{j=0}^{\overline{d}-\underline{d}} \int_{h-t_{j+1}^k}^{h-t_j^k} e^{As} ds B u_{k+j-\overline{d}} \quad (5)$$

with t_j^k as defined in Lemma 1, see also Fig. 3 for an explanation of the meaning of t_j^k. Model (5) contains each situation in (2), because each subsystem is contained in (5).

Remark: Equation (5) was also stated in [12]; however, without the explicit definition of t_j^k as in Lemma 1. In [12] it is implicitly assumed that message rejection does not occur, as $t_j^k < t_{j+1}^k$ should hold for all $0 \leq j \leq \overline{d} - \underline{d}$. Moreover, the model proposed here exhibits less uncertain parameters than the one in [12], because we consider only t_j^k as uncertain, time-varying parameters, while in [12] additional parameters are introduced that describe whether a control input is active or inactive in the sampling interval $[s_k, s_{k+1})$.

To make the model (5) suitable for the stability analysis, we rewrite it in a state-space notation, using the augmented state vector $\xi_k := (x_k^T \; u_{k-1}^T \; u_{k-2}^T \; \ldots \; u_{k-\overline{d}}^T)^T$. Then, the discrete-time NCS model is given by

$$\xi_{k+1} = \widetilde{A}(\mathbf{t}^k) \xi_k + \widetilde{B}(\mathbf{t}^k) u_k \quad (6)$$

where

$$\widetilde{A}(\mathbf{t}^k) = \begin{pmatrix} e^{Ah} & \widetilde{M}_{\overline{d}-1} & \widetilde{M}_{\overline{d}-2} & \ldots & \widetilde{M}_0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & I & 0 & \ldots & 0 \\ \vdots & & \ddots & & \ldots \\ 0 & \ldots & 0 & I & 0 \end{pmatrix}$$

$$\widetilde{B}(\mathbf{t}^k) = \begin{pmatrix} \widetilde{M}_{\overline{d}}^T & I & 0 & \ldots & 0 \end{pmatrix}^T, \quad \mathbf{t}^k = \left(t_1^k, \ldots, t_{\overline{d}-\underline{d}}^k \right)$$

$$\widetilde{M}_j = \begin{cases} \int_{h-t_{j+1}^k}^{h-t_j^k} e^{As} ds B & \text{if } 0 \leq j \leq \overline{d} - \underline{d}, \\ 0 & \text{if } \overline{d} - \underline{d} < j \leq \overline{d}. \end{cases}$$

III. STABILITY OF NCSs

To solve the stability analysis problem, we write the matrix A as $A = QJQ^{-1}$ with J the real Jordan form and Q a matrix with generalized eigenvectors, see e.g. [20]. Then, it holds that $e^{As} = Qe^{Js}Q^{-1}$. Recall that the real Jordan matrix consists of a combination of blocks, where each block corresponds to a distinct eigenvalue (or a pair of complex eigenvalues) and is given by $J = diag(J_1, J_2, \ldots, J_p)$, with $p \in \mathbb{N}$ the number of distinct (pairs of complex) eigenvalues. Each of the blocks $J_{\tilde{i}}$ consists of real Jordan blocks $J_{\tilde{i},\tilde{j}}$, $\tilde{j} = 1, 2, \ldots, g_{\tilde{i}}$ with $g_{\tilde{i}} \in \mathbb{N}$ the geometric multiplicity of the ith distinct eigenvalue (or complex eigenvalue with positive imaginary part), i.e., $J_{\tilde{i}} = diag(J_{\tilde{i},1}, J_{\tilde{i},2}, \ldots, J_{\tilde{i},g_{\tilde{i}}})$ with $\tilde{i} = 1, 2, \ldots, p$. The generic

form of the NCS model (6), based on the real Jordan form of A, including integration of the terms $\widetilde{M}_j$ in (6) is given by

$$\xi_{k+1} = \left(F_0 + \sum_{i=1}^{\nu} \sum_{j=1}^{\overline{d}-\underline{d}} \alpha_i \left(t_j^k \right) F_{i,j} \right) \xi_k$$
$$+ \left(G_0 + \sum_{i=1}^{\nu} \sum_{j=1}^{\overline{d}-\underline{d}} \alpha_i \left(t_j^k \right) G_{i,j} \right) u_k \quad (7)$$

with F_0, G_0, $F_{i,j}$, $G_{i,j}$, $i = 1, 2, \ldots, \nu$, $j = 1, 2, \ldots, \overline{d} - \underline{d}$ constant matrices that depend on Q and J and $\alpha_i(t_j^k)$ the time-varying parameters that depend on t_j^k. A typical $\alpha_i(t_j^k)$ contains terms such as $e^{\lambda(h-t_j^k)}$ and $(h - t_j^k) e^{\lambda(h-t_j^k)}$ for real eigenvalues λ and $e^{a(h-t_j^k)} \cos(b(h - t_j^k))$ and $e^{a(h-t_j^k)} \sin(b(h-t_j^k))$ for complex eigenvalues with $a \pm bj$. For more details, the reader is referred to [21]. The parameter ν is defined as $\nu = \sum_{\tilde{i}=1}^{p} q_{\tilde{i}}$ and $q_{\tilde{i}} = \max_{\tilde{j}=\{1,2,\ldots,g_{\tilde{i}}\}} (\dim J_{\tilde{i},\tilde{j}})$. See Section IV for an example.

For stability analysis, we consider the control law

$$u_k = -\overline{K} \xi_k. \quad (8)$$

Note that for the state-feedback case $u_k = -Kx_k$, on which we focus in the example in Section IV, it holds that $\overline{K} := (K \; 0_{m,\overline{d}m})$. Based on the control law (8), the NCS model of (7) is applicable for stability analysis. The (nonlinear) parameters $\alpha_i(t_j^k)$ form, together with the constant matrices $F_{i,j}$ and $G_{i,j}$, $i = 1, 2, \ldots, \nu$, $j = 1, 2, \ldots, \overline{d} - \underline{d}$, a set of matrices that describes all possible system matrices in (7)

$$\mathcal{FG} = \left\{ \left(F(\mathbf{t}^k), G(\mathbf{t}^k) \right) : \mathbf{t}^k = \left(t_1^k, \ldots, t_{\overline{d}-\underline{d}}^k \right), \right.$$
$$\left. t_j^k \in [t_{j,\min}, t_{j,\max}], 1 \leq j \leq \overline{d} - \underline{d} \right\} \quad (9)$$

with $F(\mathbf{t}^k) := F_0 + \sum_{i=1}^{\nu} \sum_{j=1}^{\overline{d}-\underline{d}} \alpha_i(t_j^k) F_{i,j}$, $G(\mathbf{t}^k) := G_0 + \sum_{i=1}^{\nu} \sum_{j=1}^{\overline{d}-\underline{d}} \alpha_i(t_j^k) G_{i,j}$ and

$$t_{j,\min} = \begin{cases} \tau_{min} - \underline{d}h, & \text{if } j = \overline{d} - \underline{d} \\ 0, & \text{if } 1 \leq j < \overline{d} - \underline{d}, \end{cases}$$
$$t_{j,\max} = \begin{cases} h, & \text{if } 1 < j \leq \overline{d} - \underline{d} \\ \tau_{max} - (\overline{d} - 1)h & \text{if } j = 1. \end{cases}$$

To guarantee the stability of the equilibrium point $\xi = 0$ of the closed loop system (7), (8), it is sufficient to prove that there exists a common quadratic Lyapunov function $V(\xi_k) = \xi_k^T P \xi_k$ for the uncertain linear system $\xi_{k+1} = (F - G\overline{K})\xi_k$, with $(F, G) \in \mathcal{FG}$ as defined in (9). This system represents a time-varying, linear discrete-time system. Hence, stability is guaranteed if the following LMIs are feasible:

$$P = P^T > 0$$
$$(F - G\overline{K})^T P (F - G\overline{K}) - P < 0, \quad \forall (F, G) \in \mathcal{FG}. \quad (10)$$

Due to the definition of $\mathcal{FG}$, an infinite number of LMIs [22] is involved in (10). In Theorem 2, we propose a stability condition based on a finite number of LMIs guaranteeing the satisfaction of (10).

Theorem 2: Consider the NCS of (1), (3), (4), (8), with delays $\tau_k \in [\tau_{\min}, \tau_{\max}]$, the corresponding discrete-time representation (6), (8) and its equivalent representation (7), (8) that is based on the Jordan form of the continuous-time system matrix A. Define the set of matrices $\mathcal{H}_{FG}$

$$\mathcal{H}_{FG} = \left\{ \left(F_0 + \sum_{i=1}^{\nu} \sum_{j=1}^{\overline{d}-\underline{d}} \alpha_{i,j} F_{i,j}, G_0 + \sum_{i=1}^{\nu} \sum_{j=1}^{\overline{d}-\underline{d}} \alpha_{i,j} G_{i,j} \right) : \right.$$
$$\left. \alpha_{i,j} \in \{\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}\}, i = 1, 2, \ldots, \nu, j = 1, 2, \ldots, \overline{d} - \underline{d} \right\} \quad (11)$$

with $\overline{\alpha}_{i,j}$:= $\max_{t_j^k \in [t_{j,\min}, t_{j,\max}]} \alpha_i(t_j^k)$, $\underline{\alpha}_{i,j}$ = $\min_{t_j^k \in [t_{j,\min}, t_{j,\max}]} \alpha_i(t_j^k)$. Note that $\mathcal{H}_{FG}$ consists of $2^{(\overline{d}-\underline{d})\nu}$ different matrices.

If there exists a matrix $P \in \mathbb{R}^{(n+\overline{d}m) \times (n+\overline{d}m)}$ such that

$$\begin{pmatrix} P & (H_F - H_G\overline{K})^T P \\ P(H_F - H_G\overline{K}) & P \end{pmatrix} > 0 \qquad (12)$$

for all $(H_F, H_G) \in \mathcal{H}_{FG}$, then (1), (3), (4), and (8) is globally asymptotically stable (GAS) for any sequence of time-varying delays $\tau_k \in [\tau_{\min}, \tau_{\max}]$. ∎

Proof: The proof is given in the Appendix. ∎

An equivalent synthesis result can be obtained easily if an extended state-feedback controller (8), with $\overline{K} \in \mathbb{R}^{m \times (\overline{d}m+n)}$ is considered. For this controller message rejection between the sensor and controller is not allowed, as new control inputs can not be computed anymore, due to the dependence on past control inputs that are not determined in the case of message rejection [21], unless some sort of buffering takes place. Note that a state-feedback controller $u_k = -Kx_k$ does not suffer from this problem. However, its synthesis problem requires $\overline{K}$ to be of a structure $(K\ 0_{m \times \overline{d}m})$, which can not be encoded easily in the LMIs below. This structured synthesis problem is the subject of future research.

Corollary 3: Let the hypotheses of Theorem 2 hold. If there exist matrices $Y \in \mathbb{R}^{(n+\overline{d}m) \times (n+\overline{d}m)}$, $Z \in \mathbb{R}^{m \times (n+\overline{d}m)}$ such that the following LMIs are satisfied:

$$\begin{pmatrix} Y & YH_F^T - Z^T H_G^T \\ H_F Y - H_G Z & Y \end{pmatrix} > 0 \qquad (13)$$

for all $(H_F, H_G) \in \mathcal{H}_{FG}$, then $\overline{K} = ZY^{-1}$ gives a control gain that renders system (1), (8) GAS and $V(\xi_k) = \xi_k^T P \xi_k$, where $P = Y^{-1}$ is a Lyapunov function for (6).

Remark: To include typical (discrete-time) Lyapunov-Krasovskii functionals (LKFs), as proposed in the literature (see e.g. [13]), in the above framework one can use (6) with the quadratic Lyapunov function $V(\xi_k) = \xi_k^T P \xi_k$ or a slightly modified form of (6) by taking the augmented state vector $\chi_k = (x_k^T\ x_{k-1}^T\ \dots\ x_{k-\overline{d}}^T)^T$ and a quadratic Lyapunov function of the form $V(\chi_k) = \chi_k^T P \chi_k$. More details on the relationship between Lyapunov functions of the form $\chi_k^T P \chi_k$ and LKFs can be found in [23].

IV. ILLUSTRATIVE MOTION CONTROL EXAMPLE

In this section, we will apply the proposed results to a second-order motion control example, obtained from the document printing domain, as in [5]. We use a single motor driving one roller that transports a paper sheet, as depicted schematically in Fig. 4. The controller is connected to the motor via a network and therefore should cope with time-varying delays. It is assumed that friction in the motor is negligible and that there is no slip between the paper sheet and the roller. The transmission between motor and roller is assumed to be rigid. The continuous-time motor-roller model is given by: $\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ qr_R/(J_M + n^2 J_R) \end{pmatrix} u$, with $x = (x_s\ \dot{x}_s)^T$ the state vector, which contains the sheet position and velocity. Moreover, $J_M = 1.95 \cdot 10^{-5}$ kgm^2 is the inertia of the motor, $J_R = 6.5 \cdot 10^{-5}$ kgm^2 is the inertia of the roller, $r_R = 14 \cdot 10^{-3}$ m is the radius of the roller, $q = 0.2$ is the transmission ratio between motor and roller and u is the motor torque. We assume that the sensor sampling interval $h = 1$ ms is constant and that the controller is given by $K = (50\ K_2)$. We determine the controller gains K_2 that stabilize the system with time-varying delays $\tau_k \in [0, \tau_{max}]$, with $\tau_{max} \leq 2h$, using Theorem 2, resulting in the gray area in Fig. 5. Here, it holds
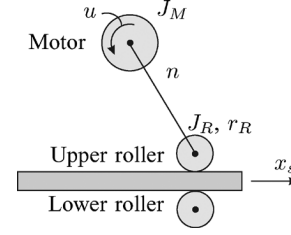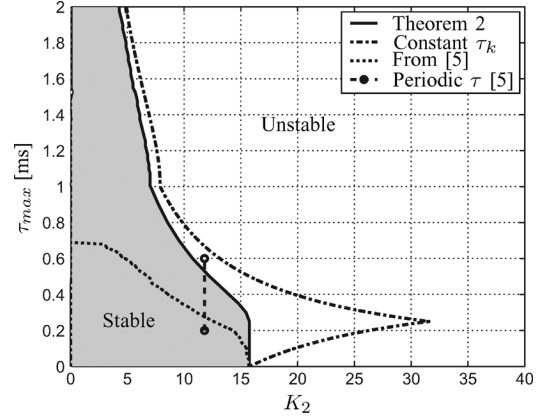

Fig. 4. Schematic overview of the motor-roller example.

Fig. 5. Stability region in terms of K_2 and time-varying delays $\tau_k \in [0, \tau_{max}]$ (for $h = 1$ ms, $K_1 = 50$) for Theorem 2, based on [5], and for constant delays equal to τ_{max}.

that $\nu = 2$, $\alpha_1(t_1^k) = h - (\tau_{k-1} - h)$, $\alpha_2(t_1^k) = (h - (\tau_{k-1} - h))^2$, $\alpha_1(t_2^k) = h - \tau_k$, and $\alpha_2(t_2^k) = (h - \tau_k)^2$. Compared to the previously published stability conditions in [5], for which the largest stabilizing K_2 is depicted by the dotted line, the results obtained by Theorem 2 are clearly less conservative. For instance, in the figure one can see that for delays in the range [0,0.6] ms the method of [5] shows stability for $0 < K_2 \leq 4.8$, while Theorem 2 proves stability for $0 < K_2 \leq 10.7$. For delays larger than 0.7 ms, the method of [5] does not provide any stabilizing controller gains, while the method proposed in this paper still gives stabilizing values for K_2. The reason for the reduction of the conservatism is the much tighter overapproximation of the discrete-time NCS model by using the real Jordan form. Moreover, the number of LMIs is much smaller. For instance, in the small delay case ($\overline{d} = 1$), [5] uses 2^{2n^2} LMIs, while here we need only 2^n LMIs, where n is the dimension of the state variable x in (1). The dashed line in Fig. 5 gives the values of two delays, i.e. $\tau_a = 0.2h$ and $\tau_b = 0.6h$ (for the controller gain $K = (50\ 11.8)$), that are both stable for constant delays and all values inbetween, but result in an unstable system for alternating delays $(\tau_a, \tau_b, \tau_a, \tau_b, \dots)$ [5]. As expected, this delay combination is outside the obtained stability region. For comparison, the stability region for *constant* time-delays equal to τ_{max} is depicted by the dash-dotted line in Fig. 5. This comparison reveals the fact that the stability bound is hardly conservative for this example, as the stability region for time-varying delays should, of course, always lie within the stability region for constant delays.

V. CONCLUSIONS

In this technical note, we presented LMI-based stability and stabilization conditions for NCSs with bounded, time-varying delays, based on a discrete-time description of the NCS. We developed a complete discrete-time and continuous-time NCS model that includes time-delays smaller and larger than the sampling interval and message rejection. This model was tightly overapproximated by a polytopic model

using the real Jordan form. In comparison with earlier work [5], the number of LMI conditions for stability was significantly decreased and the conservatism was considerably reduced. To show the applicability of the derived results, we applied them on a typical motion control example. Based on the numerical outcomes of these examples it seems that the obtained stability results are not overly conservative, which indicates the effectiveness of the results in this technical note.

APPENDIX

A. Proof of Lemma 1

From the definition of $\overline{d}$ in the lemma, we have that the control input $u_{k-\overline{d}}$ is always available before or exactly at $t = s_k := kh$ as $s_{k-\overline{d}} + \tau_{k-\overline{d}} \leq s_{k-\overline{d}} + \tau_{\max} \leq s_k$. Moreover, $u_{k-\overline{d}}$ is the oldest control input that can be active in the sampling interval $[s_k, s_{k+1})$. To prove this, consider any previous input $u_{k+j-\overline{d}}$ for some $j \leq -1$. From the definition of $\overline{d}$, we have that: $(j + k - \overline{d} + 1)h + \tau_{j+k-\overline{d}+1} \leq (j + k - \overline{d} + 1)h + \tau_{\max} \leq s_{k-\overline{d}} + \tau_{\max} \leq s_k$ for $j < 0$. This implies that the control input $u_{k+j-\overline{d}+1}, j < 0$, arrives before time s_k and thus $u_{k+j-\overline{d}+1}, j < 0$, will not be active in $[s_k, s_{k+1})$.

From the definition of $\underline{d}$ in the lemma, it follows that the input $u_{k-\underline{d}}$ represents the most recent control input that might be implemented during the sampling interval $[s_k, s_{k+1})$. Indeed, as $s_{k-\underline{d}} + \tau_{\min} < s_{k+1}$ the input $u_{k-\underline{d}}$ might be available for implementation before time s_{k+1}. To show that there is no more recent control input that might be active in the interval $[s_k, s_{k+1})$, consider the control input $u_{k+j-\overline{d}}$, for some $j > \overline{d} - \underline{d}$. From the definition of $\underline{d}$, we have that $s_{j+k-\overline{d}} + \tau_{j+k-\overline{d}} \geq s_{j+k-\overline{d}} + \tau_{\min} \geq s_{k+1}$. Therefore, the control input $u_{k+j-\overline{d}}, j > \overline{d} - \underline{d}$, can not be implemented in the sampling interval $[s_k, s_{k+1})$. Hence, the control inputs $u_{k-\overline{d}}, \ldots, u_{k-\underline{d}}$ are the only control inputs that can be active in the sampling interval $[s_k, s_{k+1})$.

The times $s_k + t_j^k, j \in \{0, \ldots, \overline{d} - \underline{d}\}$, are constructed in such a manner that $s_k + t_j^k$ is the time at which the control input $u_{k+j-\overline{d}}$ becomes active in $[s_k, s_{k+1})$. Hence, $t_{\overline{d}-\underline{d}}^k$ is given by

$$t_{\overline{d}-\underline{d}}^k = \min[h, \tau_{k-\underline{d}} - \underline{d}h]. \tag{14}$$

Indeed, if $\tau_{k-\underline{d}} - \underline{d}h \in [0, h]$, then $s_{k-\underline{d}} + \tau_{k-\underline{d}}$ is the time at which $u_{k-\underline{d}}$ is implemented. If $\tau_{k-\underline{d}} - \underline{d}h > h$, then $u_{k-\underline{d}}$ might be active after s_{k+1}. Since, we are only interested in the interval $[s_k, s_{k+1})$ we take the minimum of this value and h in (14). Note that, by definition, $\tau_{k-\underline{d}} - \underline{d}h \geq 0, \forall k$. Next, as $u_{k-\underline{d}-1}$ can only be active before $u_{k-\underline{d}}$ is available, $t_{\overline{d}-\underline{d}-1}^k$ is given by

$$t_{\overline{d}-\underline{d}-1}^k = \min\left[t_{\overline{d}-\underline{d}}^k, \max\left\{0, \tau_{k-\underline{d}-1} - (\underline{d}+1)h\right\}\right]. \tag{15}$$

Indeed, similarly to $t_{\overline{d}-\underline{d}}^k$, if $\tau_{k-\underline{d}-1} - (\underline{d}+1)h \in [0, t_{\overline{d}-\underline{d}}^k]$ then $s_k - (\underline{d}+1)h + \tau_{k-\underline{d}-1}$ is the time at which $u_{k-\underline{d}-1}$ is implemented. In case $\tau_{k-\underline{d}-1} - (\underline{d}+1)h < 0$, then $u_{k-\underline{d}-1}$ might be active before s_k. Since, we are only interested, here, in the interval $[s_k, s_{k+1})$, we take the maximum of this value and 0 in (15). For the other values of t_j^k, the recursion can be derived similarly, yielding

$$t_j^k = \min\left[t_{j+1}^k, \max\left\{0, \tau_{k+j-\overline{d}} + (j - \overline{d})h\right\}\right]$$

for $0 \leq j \leq \overline{d} - \underline{d}$, with $t_{\overline{d}-\underline{d}+1}^k := h$. Recursive substitution of these relations yields the characterization of (4).

B. Proof of Theorem 2

With $\overline{\alpha}_{i,j}$ and $\underline{\alpha}_{i,j}$ as defined in the theorem, we have that $\alpha_i(t_j^k) \in [\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}]$. Hence, the set $\overline{\mathcal{FG}}$, defined as

$$\overline{\mathcal{FG}} = \left\{ \left(F_0 + \sum_{i=1}^{\nu} \sum_{j=1}^{\overline{d}-\underline{d}} \alpha_{i,j} F_{i,j}, G_0 + \sum_{i=1}^{\nu} \sum_{j=1}^{\overline{d}-\underline{d}} \alpha_{i,j} G_{i,j} \right) : \right.$$
$$\left. \alpha_{i,j} \in [\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}], i = 1, 2, \ldots, \nu, j = 1, 2, \ldots, \overline{d} - \underline{d} \right\}$$

is an overapproximation of the set $\mathcal{FG}$ in the sense that $\mathcal{FG} \subseteq \overline{\mathcal{FG}}$. Each matrix in this set can be written as a convex combination of the generators of the set. The set of generators of $\overline{\mathcal{FG}}$ is given by $\mathcal{H}_{FG}$ in (11). The different matrices in $\mathcal{H}_{FG}$ are denoted individually by $H_{F,l}, H_{G,l}, l = 1, 2, \ldots, 2^{\nu(\overline{d}-\underline{d})}$. Based on these generators, we have for $\mathrm{co}(\mathcal{H}_{FG}) := \{(\sum_{l=1}^{2^{\nu(\overline{d}-\underline{d})}} (\phi_l H_{F,l}), \sum_{l=1}^{2^{\nu(\overline{d}-\underline{d})}} (\phi_l H_{G,l})) : \sum_{l=1}^{2^{\nu(\overline{d}-\underline{d})}} \phi_l = 1, \text{with } \phi_l \geq 0, l = 1, 2, \ldots, 2^{\nu(\overline{d}-\underline{d})}\}$ that

$$\mathcal{FG} \subseteq \overline{\mathcal{FG}} = \mathrm{co}(\mathcal{H}_{FG}). \tag{16}$$

Next, we show that (12) is sufficient to guarantee the satisfaction of (10). Since (12) holds for all $(H_F, H_G) \in \mathcal{H}_{FG}$, we have that, by using the Schur complement

$$\begin{pmatrix} P & (H_{F,l} - H_{G,l}\overline{K})^T P \\ P(H_{F,l} - H_{G,l}\overline{K}) & P \end{pmatrix} > 0 \tag{17}$$

for all $l \in \{1, 2, \ldots, 2^{\nu(\overline{d}-\underline{d})}\}$. Multiplying (17) for each l by $\phi_l \geq 0$, summing them and using that $\sum_{l=1}^{2^{\nu(\overline{d}-\underline{d})}} \phi_l = 1$ gives

$$\begin{pmatrix} P & (\overline{H_F} - \overline{H_G K})^T P \\ P(\overline{H_F} - \overline{H_G K}) & P \end{pmatrix} > 0, \tag{18}$$

for all $(\overline{H_F}, \overline{H_G}) \in \mathrm{co}(\mathcal{H}_{FG})$. Due to (16), (18) implies for all $(F, G) \in \mathcal{FG}$ that

$$\begin{pmatrix} P & (F - G\overline{K})^T P \\ P(F - G\overline{K}) & P \end{pmatrix} > 0.$$

Applying the Schur complement again gives (10), which shows that $V(\xi_k) = \xi_k^T P \xi_k$ is a Lyapunov function for (7), (8) that proves GAS of the origin $\xi = 0$ of (6), (8).

Based on the reasoning in [5], this also includes the intersample behavior and therefore GAS of $x = 0$ for the continuous-time system (1), (8).

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Relaxed Conditions for the Exponential Stability of a Class of Linear Time-Varying Systems

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Abstract—This technical note states new sufficient conditions for the exponential stability of linear time-varying (LTV) systems of the form $\dot{x}(\cdot) = A(t)x(t)$. The approach proposed derives and uses the notion of perturbed frozen time (PFT) form that can be associated to any LTV system. Exploiting the Bellman-Gronwall lemma, relaxed stability conditions are then stated in terms of "average" parameter variations. Salient features of the approach are: pointwise stability of $A(\cdot)$ is not required, $\|\dot{A}(\cdot)\|$ may not be bounded, the stability conditions also apply to uncertain systems. The approach is illustrated by numerical examples.

Index Terms—Linear systems, stability conditions, time-varying systems.

I. INTRODUCTION

The purpose of this technical note is to state new sufficient stability conditions for a LTV system Σ described by

$$\dot{x}(t) = A(t)x(t), \quad x(0) = x_0, \quad t \geq 0. \tag{1}$$

Many authors investigated this problem using the frozen-time approach (FTA), whose main advantage is the possibility of exploiting the great deal of tools which have been developed for linear time-invariant (LTI) systems. The first papers dealing with this topic showed that pointwise stability implies stability of the LTV system provided that $\sup_{t \geq 0} \|\dot{A}(\cdot)\| < \delta$ for sufficiently small δ [1], [2]. The pointwise stability is also required in [3], [11] which extends previous results [4]–[6], to derive explicit upper bounds for different measures of parameter variations guaranteeing stability. Under a slightly weaker assumption on pointwise stability, the FTA approach has been also used in [7] to derive sufficient stability conditions both for continuous and discrete-time LTV systems. Pointwise stability has been also recently exploited in [8], where the stability analysis is performed solving successive Lyapunov equations defined on a time grid. In [9], sufficient stability conditions are derived requiring that the eigenvalues of $A(t)$ be stable "on average" for $t \geq 0$.

The approach developed in this technical note is based on the notion of perturbed frozen time (PFT) form of a LTV system and uses the continuous-time version of the Bellman-Gronwall lemma [10]. The system is not required to be pointwise stable or slowly varying and the dynamical operator $A(\cdot)$ is not required to be differentiable. The relaxed sufficient stability conditions are derived here assuming that there exists a known sequence of time instants at which the corresponding frozen time plant is stable. Between any two consecutive time instants, quick and/or large parametric variations with respect to the frozen plant are allowed, provided that the "average" variation is small enough.

The salient features of this technical note are: 1) pointwise stability is not required; 2) the plant is not required to be slowly varying, namely no bound is imposed on $\|\dot{A}(\cdot)\|$ (provided $\dot{A}(\cdot)$ exists); 3) the stability conditions are easy to be checked; 4) the method also applies in the