Numerical Evaluation of a Robust Self-Triggered MPC Algorithm [★]

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Abstract: We present numerical examples demonstrating the efficacy of a recently proposed self-triggered model predictive control scheme for disturbed linear discrete-time systems with hard constraints on the input and state. In order to reduce the amount of communication between the controller and the actuator, the control input is not re-computed at each point in time but only at certain sampling instances. These instances are determined in a self-triggered fashion in the sense that at every sampling instant the next sampling instant is computed as a function of the current system state. A compact set in the state space, whose size is a design parameter in the control scheme, is stabilized.

Keywords: predictive control, self-triggered control, robustness

1. INTRODUCTION

In networked control systems, the rate of communication between the different components of the control loop plays an important role in the overall performance of the system. On the one hand, higher sampling rates generally lead to better disturbance rejection, as the system can react faster to changes in the output. On the other hand, higher sampling rates increase the energy consumption in the communication channel, which is especially pronounced in wireless communication; on top of that, the bandwidth of the communication network might be limited, which is in particular the case if the network has to be shared among multiple components, see, for example, Hespanha et al. (2007).

It has been found that aperiodic sampling schemes, where the times between sampling instants depend on the evolution of the system state, lead to a better trade-off between the controller performance and the required average communication rate than strictly periodic sampling schemes. See, for example, Åström and Bernhardsson (2002); Antunes and Heemels (2014), where this was shown quantitatively. One approach to aperiodic sampling is self-triggered control. Here, the next sampling instant is explicitly computed as a function of the available information at the current sampling instant. This allows the sensor, and potentially even the communication network, to be completely shut down between sampling instances, saving additional energy on top of the energy saved by not transmitting information over the network. For a recent overview of self-triggered control, see, for example, Heemels et al. (2012).

In this paper, we consider the control of linear discrete-time systems subject to additive bounded disturbances and hard constraints on the input and state. Model predictive control (MPC) has been found to be a control method well suited for constrained control problems, see, for example, Rawlings and Mayne (2009). For additively perturbed systems, robust MPC methods have to be applied, one of which is Tube MPC, see Mayne et al. (2005); Chisci et al. (2001). In MPC, a finite horizon optimal control problem depending on the current system state is solved at every time step and the first element of the resulting optimal input sequence is applied to the system. In order to guarantee robust constraint satisfaction, worst-case set-valued predictions of the future system state are incorporated in the optimal control problems in Tube MPC. An important development in the field was the assumption of feedback at every future point in the predictions, which prevents exponential growth of the uncertainties in the predicted system state, see Chisci et al. (2001). In the context of self-triggered control, however, the assumption of feedback at *every* point in time is not justified, necessitating the development of appropriately modified approaches. One such approach was proposed in Brunner et al. (2014), with the drawback that the parameters of the whole scheme proposed there depend on the *maximally allowed* time between sampling instances, which might limit the region of attraction, a problem that was partially addressed in Aydiner et al. (2015). In the present paper, we include the time M between the current sampling instant and the next as a decision variable in the optimal control problem. Following the ideas in Gommans and Heemels (2015), at each sampling instant we select the largest M

for which the resulting optimal cost function is lower than the optimal cost function for $M = 1$, allowing the cost function associated with $M = 1$ to be used as Lypunov-type function for the closed-loop system.

Alternative self-triggered MPC schemes for discrete-time systems, with and without disturbances, can, for example, be found in Kögel and Findeisen (2015), Gommans and Heemels (2015), Gommans (2016), Eqtami (2013), and the references therein.

The technical aspects of the control scheme presented here were already discussed in Brunner et al. (2016). The contributions of the present paper are the numerical examples provided in Section 4. In Brunner et al. (2016), no examples were given.

The remainder of the paper is structured in the following way. This introductory section concludes with a few remarks on notation. The problem setup is presented in Section 2. In Section 3, the robust self-triggered MPC algorithm is presented. Numerical examples highlighting different aspects of the approach are given in Section 4. Section 5 concludes the paper.

Notation: Let $\mathbb{N}$ denote the set of non-negative integers. For $q, s \in \mathbb{N} \cup \{\infty\}$, let $\mathbb{N}_{\geq q}$ and $\mathbb{N}_{[q,s]}$ denote the sets $\{r \in \mathbb{N} \mid r \geq q\}$ and $\{r \in \mathbb{N} \mid q \leq r \leq s\}$, respectively. The set of non-negative real numbers is denoted by $\mathbb{R}_+$. Given sets $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^n$, a scalar α, and a matrix $A \in \mathbb{R}^{m \times n}$, we define $\alpha\mathcal{X} := \{\alpha x \mid x \in \mathcal{X}\}$ and $A\mathcal{X} := \{Ax \mid x \in \mathcal{X}\}$ The Minkowski set addition is defined by $\mathcal{X} \oplus \mathcal{Y} := \{x + y \mid x \in \mathcal{X}, y \in \mathcal{Y}\}$. For any vectors $v_i \in \mathbb{R}^n$, $i \in \mathbb{N}$, we define $\sum_{i=a}^b v_i = 0$ for any $a, b \in \mathbb{N}$ if $a > b$. We call a compact, convex set containing the origin a C-set. A C-set containing the origin in its (non-empty) interior is called a PC-set. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}_\infty$ if it is continuous, strictly increasing, $\alpha(0) = 0$ and $\alpha(s) \to \infty$ as $s \to \infty$. The Euclidean norm of a vector $v \in \mathbb{R}^n$ is denoted by $|v|$. Given any compact set $\mathcal{S} \subseteq \mathbb{R}^n$, the distance between v and $\mathcal{S}$ is defined by $|v|_\mathcal{S} := \min_{s \in \mathcal{S}} |v - s|$. Define finally the Euclidean unit ball by $\mathcal{B} := \{x \in \mathbb{R}^n \mid |x| \leq 1\}$.

2. PROBLEM SETUP

The problem setup is taken from Brunner et al. (2016).

We consider discrete-time linear time-invariant systems subject to bounded additive disturbances given by

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, and $w_k \in \mathcal{W} \subseteq \mathbb{R}^n$ denote the state, control input, and unknown disturbance, respectively, at discrete time $k \in \mathbb{N}$. Moreover, $\mathcal{W}$ is a known C-set. The state x_k is available as a measurement.

Our goal is to robustly asymptotically stabilize a set containing the origin of (1) while satisfying the constraints $x_k \in \mathcal{X}$ and $u_k \in \mathcal{U}$ for all $k \in \mathbb{N}$, where $\mathcal{X}$ and $\mathcal{U}$ are PC-sets. We want to achieve this goal by implementing a control law for which only sporadic measurements of the system state are necessary, thereby reducing the overall communication load in the control system. Furthermore, we want to guarantee an upper bound for the closed-loop performance that is not worse than the upper bound guaranteed by a periodically updated MPC scheme, multiplied by a positive factor of our choice.

In the closed-loop system, the input is given by

$$u_k = \kappa(x_{k_j}, k - k_j), \ k \in \mathbb{N}_{[k_j, k_{j+1}-1]}, \ j \in \mathbb{N}, \tag{2}$$

for a function $\kappa : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^m$. The control actions and state measurements are only updated at the time points $k_j \in \mathbb{N}$, called sampling instants henceforth. These sampling instants shall be determined in a self-triggered fashion, that is,

$$k_0 = 0, \ k_{j+1} = k_j + \mu(x_{k_j}), \ j \in \mathbb{N}, \tag{3}$$

for a function $\mu : \mathbb{R}^n \to \mathbb{N}_{\geq 1}$. Hence, the problem addressed in this paper is finding functions κ and μ such that the closed-loop system (1)-(3) exhibits the properties mentioned above.

Remark 1. We allow the input to be time-varying between sampling instants. However, the input is open-loop in the sense that it is only allowed to depend on the state at the last sampling instant. If the definition of the feedback law in (2) is changed to $u_k = \kappa(x_{k_j}, 0)$, $k \in \mathbb{N}_{[k_j, k_{j+1}-1]}$, $j \in \mathbb{N}$, then the input is constant between sampling instants, which further reduces the amount of data communicated over the controller-to-actuator channel. Changing the requirement to $u_k = \kappa(x_{k_j}, 0)$, $k \in \{k_j \mid j \in \mathbb{N}\}, u_k = 0$, otherwise, promotes sparsity in the input signal in addition to reducing the amount of communication. Similarly, the aforementioned scheme with constant inputs between sampling instants promotes sparsity in the difference between input values at subsequent points in time. Please refer to Gommans and Heemels (2015), section 3.1 and 3.2, for an extended discussion of this matter.

We make use of a stabilizing linear state-feedback controller $u = Kx$ in the paper. The following assumption holds.

Assumption 2. The eigenvalues of the matrix $A + BK$ are contained in the interior of the complex unit disc.

3. ROBUST SELF-TRIGGERED MPC

In this section, we present a solution to the problem stated in Section 2 based on robust model predictive control. The main idea is to include the time M until the next sampling instant as a decision variable in the optimization problem. For details, please refer to Brunner et al. (2016).

3.1 Constraints

Let the decision variable of the finite-horizon optimal control problem at time point k be given by

$$\mathbf{d}_k = ((x_{0|k}, \dots, x_{N|k}), (u_{0|k}, \dots, u_{N-1|k})) \in \mathbb{D}_N, \tag{4}$$

where $\mathbb{D}_N = \mathbb{R}^n \times \cdots \times \mathbb{R}^n \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m$ and $N \in \mathbb{N}_{\geq 1}$ is the prediction horizon.

Depending on the number $M \in \mathbb{N}_{[1,N]}$ of open-loop steps until the next sampling instant, different constraints are imposed on $\mathbf{d}_k$. In particular, for a given system state x_k at time point k we impose the constraints

$$x_{0|k} = x_k, \tag{5a}$$

$$\forall i \in \mathbb{N}_{[0,N-1]}, \ x_{i+1|k} = Ax_{i|k} + Bu_{i|k}, \tag{5b}$$

$$\forall i \in \mathbb{N}_{[0,N-1]}, \quad x_{i|k} \in \mathcal{X}_i^M, \tag{5c}$$

$$\forall i \in \mathbb{N}_{[0,N-1]}, \quad u_{i|k} \in \mathcal{U}_i^M, \tag{5d}$$

$$x_{N|k} \in \mathcal{X}_\mathrm{f}^M \tag{5e}$$

on the decision variable $\mathbf{d}_k$, where the variables $x_{i|k}$ represent a predicted trajectory for the undisturbed system

generated by the inputs $u_{i|k}$. The sets $\mathcal{X}_i^M$ and $\mathcal{U}_i^M$, $i \in \mathbb{N}_{[0,N-1]}$, are tightened constraint sets, each depending on the step i in the prediction and the number M of open-loop steps. The set $\mathcal{X}_f^M$ is a terminal set. Define the set of all feasible decision variables for a given point $x_k \in \mathbb{R}^n$ and a fixed M by

$$\mathcal{D}_N^M(x_k) = \{\mathbf{d}_k \in \mathbb{D}_N \mid \text{(5a) to (5e)}\}. \qquad (6)$$

3.2 Cost Function

For a system affected by bounded disturbances, a sensible stability notion is the stability of a set $\mathcal{Y} \subseteq \mathbb{R}^n$ which is robust positively invariant (RPI) under a certain state feedback law, (compare Kolmanovsky and Gilbert (1998)). We expect the size of this set to be traded off with the average inter-sampling time in the closed-loop system. In order to make this trade-off accessible in the design phase, the set $\mathcal{Y}$ is chosen to be a parameter in the MPC scheme. For simplicity, we choose $\mathcal{Y}$ to be an RPI set for system (1) in closed-loop with the feedback law $u_k = Kx_k$ (see Assumption 2 above). That is, we assume that $(A + BK)\mathcal{Y} \oplus \mathcal{W} \subseteq \mathcal{Y}$. Both the performance specification and the cost function are defined in terms of this set. In particular, we consider the infinite-horizon performance index

$$V_\infty(x_0) := \sum_{k=0}^\infty \min_{\substack{y_k \in \mathcal{Y} \\ v_k \in K\mathcal{Y}}} \ell(x_k - y_k, u_k - v_k) \qquad (7)$$

for system (1) in closed-loop with the self-triggered controller and initial condition x_0, with the stage cost function $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$. Define the stage cost functions $\bar{\ell}_i^M : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $i \in \mathbb{N}_{[0,N-1]}$, and the terminal cost functions $\bar{V}_f^M : \mathbb{R}^n \to \mathbb{R}$, each for $M \in \mathbb{N}_{[1,N]}$, where

$$\bar{\ell}_i^M(x,u) := \begin{cases} \min\limits_{\substack{y \in \mathcal{Y}_i^M \\ v \in \mathcal{V}_i^M}} \max\limits_{e \in \mathcal{E}_i^M} \ell(x - y + e, u - v) & \\ & i \in \mathbb{N}_{[0,M-1]} \\ \min\limits_{y \in \mathcal{Y}_i^M} \max\limits_{e \in \mathcal{E}_i^M} \ell(x - y + e, u - Ky + Ke) & \\ & i \in \mathbb{N}_{[M,N-1]} \end{cases}$$
$$(8a)$$

$$\bar{V}_f^M(x) := \min_{y \in \mathcal{Y}_N^M} \max_{e \in \mathcal{E}_N^M} V_f(x - y + e), \qquad (8b)$$

for $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Compare Kerrigan and Maciejowski (2004) for cost functions similar to (7) and the same paper and references therein for MPC approaches involving min-max optimization.

An appropriate choice of the sets $\mathcal{Y}_i^M \subseteq \mathbb{R}^n$, $i \in \mathbb{N}_{[0,N]}$, $\mathcal{V}_i^M$, $i \in \mathbb{N}_{[0,M-1]}$, and $\mathcal{E}_i^M$, $i \in \mathbb{N}_{[0,N]}$, ensures that the set $\mathcal{Y}$ is asymptotically stable for the closed-loop system. The stage cost function $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and the terminal cost function $V_f : \mathbb{R}^n \to \mathbb{R}$ satisfy the standard assumptions $\ell(x,u) \geq \alpha_1(|x|)$, $0 \leq V_f(x) \leq \alpha_2(|x|)$, and $V_f((A + BK)x) \leq V_f(x) - \ell(x)$ for all $x \in \mathbb{R}^n$, all $u \in \mathbb{R}^m$ and some $\mathcal{K}_\infty$-functions α_1 and α_2. See Brunner et al. (2016) for definitions and properties of the sets $\mathcal{Y}_i^M$, $\mathcal{V}_i^M$, and $\mathcal{E}_i^M$.

The M-step open-loop finite horizon problem at a state x_k then reads

$$\bar{V}_N^M(x_k) := \min_{\mathbf{d}_k \in \mathcal{D}_N^M(x_k)} \bar{J}_N^M(\mathbf{d}_k), \qquad (9)$$

where

$$\bar{J}_N^M(\mathbf{d}_k) = \sum_{i=0}^{M-1} \frac{1}{\beta} \bar{\ell}_i^M(x_{i|k}, u_{i|k})$$
$$+ \sum_{i=M}^{N-1} \bar{\ell}_i^M(x_{i|k}, u_{i|k}) + \bar{V}_f^M(x_{N|k}), \qquad (10)$$

which is inspired by the cost function proposed in Barradas Berglind et al. (2012). The parameter $\beta \geq 1$ allows a trade-off between the performance (in terms of the infinite horizon cost function) and the average sampling rate, see Barradas Berglind et al. (2012); Gommans et al. (2014).

3.3 Control Algorithm

In this section, we propose a solution to the problem statement in Section 2 in the form of an MPC controller which maximizes the number of steps until the next control update, subject to certain conditions that will guarantee robust constraint satisfaction, stability, and performance properties. In particular, for any $k \in \mathbb{N}$ and any $x_k \in \mathbb{R}^n$ define the optimization problem

$$M^\star(x_k) := \max \Big\{ M \in \mathbb{N}_{[1,M_{\max}]} \Big| \mathcal{D}_N^1(x_k) \neq \emptyset,$$
$$\mathcal{D}_N^M(x_k) \neq \emptyset, \ \bar{V}_N^M(x_k) \leq \bar{V}_N^1(x_k) \Big\} \quad (11a)$$

$$\mathbf{d}_k^\star(x_k) := \operatorname*{argmin}_{\mathbf{d}_k^M \in \mathcal{D}_N^{M^\star(x_k)}(x_k)} \bar{J}_N^{M^\star(x_k)}(\mathbf{d}_k) \qquad (11b)$$

with $\mathbf{d}_k^\star(x_k) = ((x_{0|k}^\star(x_k), \ldots, x_{N|k}^\star(x_k)), (u_{0|k}^\star(x_k), \ldots, u_{N-1|k}^\star(x_k)))$ where $M_{\max} \in \mathbb{N}_{[1,N]}$ is a chosen maximal length of the open-loop phase. The optimization problem in (11) forms the basis of the following control algorithm.

Algorithm 1 Self-Triggered Tube MPC

1: Set $k = 0$.
2: At time k, obtain the current state x_k of system (1).
3: Solve the optimization problems in (11), obtain $M^\star(x_k)$ and $\mathbf{d}_k^\star(x_k)$.
4: Apply $u_{k+i} = u_{i|k}^\star(x_k)$ to the system for $i \in \mathbb{N}_{[0,M^\star(x_k)-1]}$.
5: At time $k + M^\star(x_k)$, set $k = k + M^\star(x_k)$.
6: Go to 2.

The set of states where Algorithm 1 is feasible is $\hat{\mathcal{X}}_N = \{x \in \mathbb{R}^n \mid \mathcal{D}_N^1(x) \neq \emptyset\}$. The closed-loop system resulting from the application of Algorithm 1 is

$$x_{k+1} = Ax_k + Bu_k + w_k$$
$$\text{where } u_k = \kappa(x_{k_j}, k - k_j) \text{ if } k \in \mathbb{N}_{[k_j, k_{j+1}-1]} \quad (12a)$$
$$k_{j+1} = k_j + \mu(x_{k_j}), \qquad (12b)$$

for $j \in \mathbb{N}$, $k_0 = 0$, $x_0 \in \hat{\mathcal{X}}_N$, and $w_k \in \mathcal{W}$ for all $k \in \mathbb{N}$, where the functions κ and μ are given by

$$\kappa(x_{k_j}, k - k_j) := u_{k-k_j|k_j}^\star(x_{k_j}) \text{ if } k \in \mathbb{N}_{[k_j, k_{j+1}-1]} \quad (13a)$$
$$\mu(x_{k_j}) := M^\star(x_{k_j}). \qquad (13b)$$

Theorem 3. (Recursive feasibility). For all $x_0 \in \hat{\mathcal{X}}_N$, the closed-loop system (12) is well defined, that is, if $x_0 \in \hat{\mathcal{X}}_N$, then for all $j \in \mathbb{N}$ and all k_j the optimization problem in (11) admits a solution for x_{k_j}. Furthermore, if $x_0 \in \hat{\mathcal{X}}_N$, then for all $k \in \mathbb{N}$ and all $j \in \mathbb{N}$ it holds that $x_k \in \mathcal{X}$ and $u_k \in \mathcal{U}$ for any disturbances $w_k \in \mathcal{W}$, $k \in \mathbb{N}$.

Theorem 4. (Performance Bound). For any $x_0 \in \hat{\mathcal{X}}_N$ and any disturbances $w_k \in \mathcal{W}$, $k \in \mathbb{N}$, the closed-loop dynamics (12) satisfy the performance bound

$$\sum_{k=0}^{\infty} \min_{\substack{y_k \in \mathcal{Y} \\ v_k \in K\mathcal{Y}}} \ell(x_k - y_k, u_k - v_k) \leq \beta \bar{V}_N^1(x_0) \qquad (14)$$

defined in terms of the stage cost function ℓ, the set $\mathcal{Y}$, the parameter β, and the optimal cost function $\bar{V}_N^1$ for the MPC scheme were sampling is assumed to occur at every point in time.

Theorem 5. (Asymptotic Bound). For the closed-loop dynamics (12) and any $x_0 \in \hat{\mathcal{X}}_N$ it holds that x_k converges to the set $\mathcal{Y}$ as k approaches infinity in the sense that $\lim_{k\to\infty} |x_k|_{\mathcal{Y}} = 0$ for any disturbances with $w_k \in \mathcal{W}$, $k \in \mathbb{N}$.

Theorem 6. (Robust Asymptotic Stability). Let $\tilde{\mathcal{X}}_N \subseteq \mathbb{R}^n$ be the set of states where the MPC optimization problem for $M = 1$ is feasible with $u_{i|k} = Kx_{i|k}, i \in \mathbb{N}_{[0,N-1]}$. If there exists an $\eta > 0$ such that $\eta\mathcal{B} \oplus \mathcal{Y} \subseteq \tilde{\mathcal{X}}_N$, then the set $\mathcal{Y}$ is robustly asymptotically stable for the closed-loop dynamics (12) and the set $\hat{\mathcal{X}}_N$ belongs to its region of attraction.

4. NUMERICAL EXAMPLES

Consider the system

$$x_{k+1} = \begin{bmatrix} 1.1 & 1 \\ 0 & 1.2 \end{bmatrix} x_k + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u_k + w_k, \qquad (15)$$

where for all $k \in \mathbb{N}$, $w_k \in \mathcal{W}$ with $\mathcal{W} = [-0.25, 0.25] \times [-0.25, 0.25]$. The state and input constraints are given by $\mathcal{X} = [-500, 500] \times [-500, 500]$ and $\mathcal{U} = [-20, 20]$. The stage and terminal cost functions are given by $\ell(x, u) = x^\top Q x + u^\top R u$ and $V_f(x) = x^\top P x$ for all $x \in \mathbb{R}^2$ and all $u \in \mathbb{R}$, where $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R = 0.1$, and P is the stabilizing and optimal solution to the discrete time algebraic Riccati equation $A^\top X A - X - A^\top X B (B^\top X B + R)^{-1} B^\top X A + Q = 0$. The feedback matrix K is chosen to $K = -(B^\top X B + R)^{-1} B^\top X A$. Furthermore, we calculated $\bar{\mathcal{Y}}$ as an RPI outer approximation of the minimal RPI set for the dynamics $x^+ = (A + BK)x + w$ with $w \in \mathcal{W}$ according to Raković et al. (2005), with the approximation tolerance chosen to $\epsilon = 0.1$. The set $\mathcal{Y}$ was calculated as [1] $\mathcal{Y} := c_1\bar{\mathcal{Y}}$ for different values of c_1. For the computation of the sets $\mathcal{Y}_i^M$ and $\mathcal{E}_i^M$ please refer to Brunner et al. (2016). In particular, the sets $\mathcal{E}_i^M$ have been calculated as proposed in section 6.3 of Brunner et al. (2016) with $\mathcal{E} = [-1, 1] \times [-1, 1]$. The prediction horizon has been chosen to $N = 20$, the maximal open-loop length to $M_{\max} = 8$. An additional constraint, that is $u_{i|k} = u_{0|k}$ for all $i \in \mathbb{N}_{[1,M-1]}$, has been added to the constraint set $\mathcal{D}_N^M$ in order to promote sparsity in the difference of the input values between subsequent time steps, as described in Remark 1. For different values of c_1, the sets $\bar{\mathcal{Y}}$, $\tilde{\mathcal{X}}_N$, and $\mathcal{Y}$ are depicted in Figure 1, the set $\tilde{\mathcal{X}}_N$ being the set of states where the MPC optimization problem for $M = 1$ is feasible with $u_{i|k} = Kx_{i|k}, i \in \mathbb{N}_{[0,M-1]}$.

4.1 Dependence on the size of $\mathcal{Y}$

In order to investigate the dependence of the limit behavior of the closed-loop system on the size of $\mathcal{Y}$, simulations

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[1] Note that if $\bar{\mathcal{Y}}$ is RPI for the dynamics $x^+ = (A + BK)x + w$, $w \in \mathcal{W}$, with a C-set $\mathcal{W}$, so is $c_1\bar{\mathcal{Y}}$ if $c_1 \geq 1$.

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have been performed with initial condition $x_0 = (0, 0)^\top$, a constant disturbance $w_k = (0.25, 0.25)^\top$, $k \in \mathbb{N}$, and a simulation length of $T_{\text{sim}} = 500$ steps. In all simulations in this section, the tolerance for checking the inequality in (11) has been set to 10^{-9}.
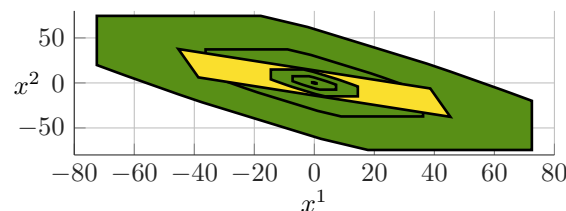

Fig. 1. The sets $c_1\bar{\mathcal{Y}}$, used as set $\mathcal{Y}$ in the simulations, for $c_1 \in \{1, 10, 20, 50, 100\}$. The set $\tilde{\mathcal{X}}_N$ is depicted for comparison. For $c = 20$, it still holds that $c\bar{\mathcal{Y}}$ is contained in the interior of $\tilde{\mathcal{X}}_N$.
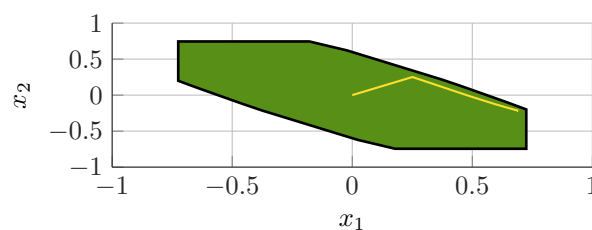

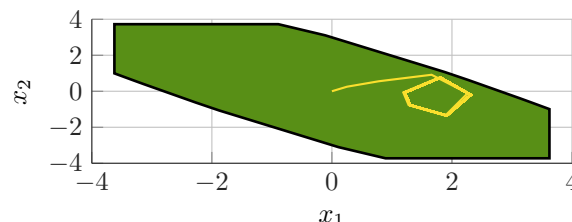Fig. 2. Closed-loop trajectory for $c_1 = 1$ and the set $c_1\bar{\mathcal{Y}}$.

Fig. 3. Closed-loop trajectory for $c_1 = 5$ and the set $c_1\bar{\mathcal{Y}}$.

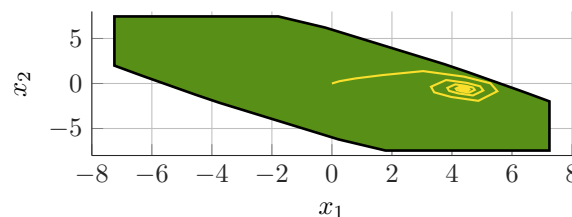

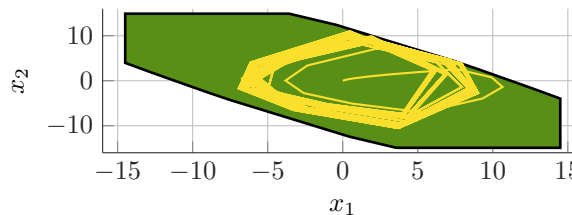Fig. 4. Closed-loop trajectory for $c_1 = 10$ and the set $c_1\bar{\mathcal{Y}}$.

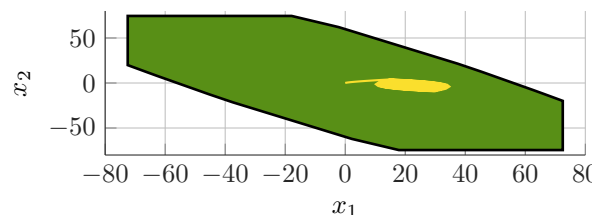Fig. 5. Closed-loop trajectory for $c_1 = 20$ and the set $c_1\bar{\mathcal{Y}}$.

Fig. 6. Closed-loop trajectory for $c_1 = 100$ and the set $c_1\bar{\mathcal{Y}}$.

The average number of steps between sampling instants for these simulations are given in Table 1. Note that the parameter β has no effect for these simulations, as an initial condition of $x_0 = (0,0)^\top$ implies that the cost function $\bar{V}_N^1$ is zero at each time step. Closed-loop trajectories for different values of c_1 and the associated sampling behavior are shown in Figure 2 to Figure 11. It is particularly noteworthy that the dependence of the average sampling frequency of c_1 is non-monotonous (see Table 1) and that sampling with non-constant frequency
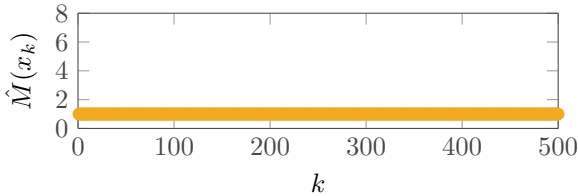

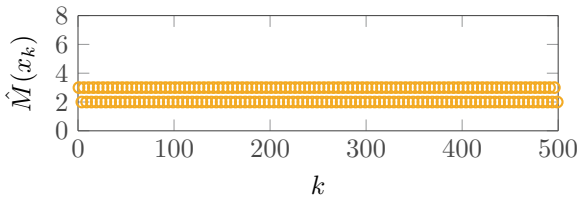Fig. 7. Sampling behavior for $c_1 = 1$.

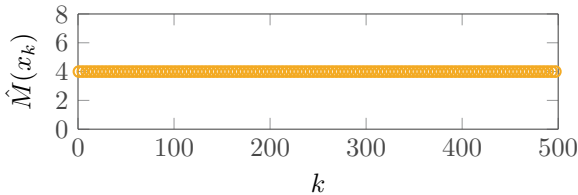Fig. 8. Sampling behavior for $c_1 = 5$.

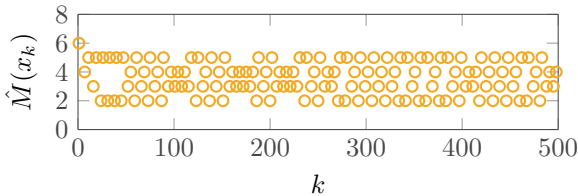Fig. 9. Sampling behavior for $c_1 = 10$.
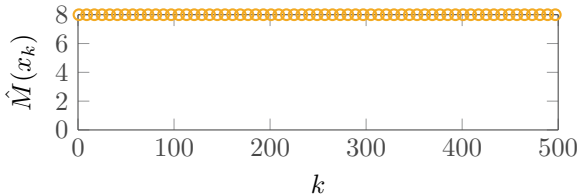

Fig. 10. Sampling behavior for $c_1 = 20$.

Fig. 11. Sampling behavior for $c_1 = 100$.

Table 1. Relation between average inter-sampling times and the asymptotic bound.

c_1	average inter - sampling time	c_1	average inter - sampling time
1.0	1.00	25.0	3.62
3.0	2.00	40.0	3.97
3.5	2.00	50.0	4.39
5.0	2.50	60.0	4.90
10.0	4.00	70.0	5.38
13.0	3.50	80.0	5.95
15.0	3.60	90.0	6.76
20.0	3.55	100.0	7.94

appears both for low and high values of c_1. While the exact interdependence of the triggering behavior with the parameters of the MPC scheme is an open question, it seems that higher sampling frequencies coincide with limit-cycles close to the boundary of the set $\bar{\mathcal{Y}}$, compare, for example, the closed-loop trajectory for $c_1 = 10$ in Figure 4 with the closed-loop trajectory for $c_1 = 20$ in Figure 5 and observe that the sampling frequency is higher for $c = 20$. Another reason for the non-monotonicity might be related to the fact the the self-triggered approach is "greedy" in the sense that only the time until the very next sampling instant is maximized, and not the average inter-sampling time over an infinite horizon. See also Gommans et al. (2014) for a more detailed discussion of this matter.

4.2 Dependence on the disturbance sequence

It is important to note that the closed-loop behavior strongly depends on the particular realization of the disturbance sequence, which makes it difficult to give general statements. The next example illustrates this point. Consider the same setup as above. Figure 12 depicts the closed-loop trajectories for $c_2 = 20$ and the disturbance

$$\forall k \in \mathbb{N}, \; w_k = \begin{cases} (0.25, 0.25)^\top & \exists j \in \mathbb{N} : k = 3j \\ (0,0)^\top & \text{otherwise,} \end{cases} \quad (16)$$

such that w_k is only non-zero at every third step. As it turns out, the number of steps between sampling instants is constantly 6 for this simulation, which is a lot greater than the average inter-sampling time for $c_1 = 20$ and a constant disturbance $w_k = (0.25, 0.25)^\top, k \in \mathbb{N}$, which was 3.55, compare Table 1. As the sequence in (16) corresponds to a sporadically occurring disturbance, the lower sampling frequency can be explained by the plant requiring less action from the controller under these circumstances. This shows that the self-triggered control scheme is able to take advantage of below-worst-case disturbances, which is not possible in a periodically updated scheme with fixed sampling rate.
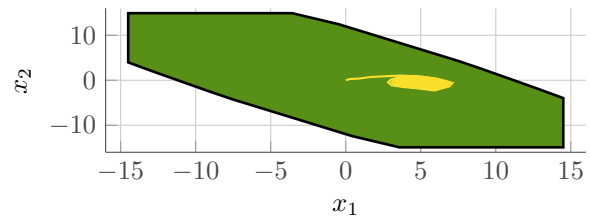

Fig. 12. Closed-loop trajectory for $c_1 = 20$ and a disturbance that is nonzero only at every third time step. Sampling occurs at exactly every 6th time step. The set $c_1\bar{\mathcal{Y}}$ is also depicted.
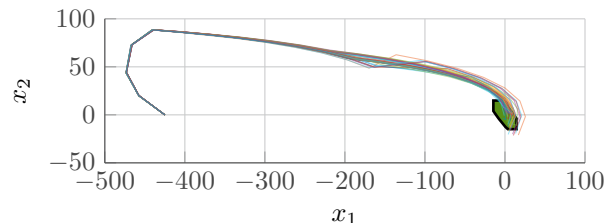

Fig. 13. Closed-loop trajectories for 50 randomly generated disturbance sequences, $\beta = 1.75$, $c_1 = 20$, starting at the same initial condition $x_0 = (-425, 0)^\top$. The set $c_1\bar{\mathcal{Y}}$ is also depicted.

Table 2. Relation between average inter-sampling times and performance.

β	average inter-sampling time	$J_{\text{perf}}/10^6$	$\beta\bar{V}_N^1(x_0)/10^6$
1.00	1.74	1.588	3.228
1.25	3.49	1.619	3.993
1.50	3.72	1.630	4.757
1.75	3.96	1.635	5.522
2.00	3.96	1.636	6.287
2.25	4.22	1.678	7.052
2.50	4.77	1.705	7.817
2.75	5.25	1.743	8.581
3.00	5.25	1.759	9.346

4.3 Performance trade-off

In order to investigate the trade-off between performance and the sampling frequency, closed-loop simulations for different values of the parameter β have been performed with initial condition $x_0 = (-425, 0)^\top$ and $c_1 = 20$. The simulation was performed only over $T_{\text{sim}} = 21$ steps, as the sampling frequency for large times is determined by the limit behavior investigated above. The disturbance sequence has been generated by sampling uniformly on $\mathcal{W}$, independently at each time step. For each value of β, the average over 50 different random disturbance sequences was taken. We used the same disturbance sequences for every value of β. In Table 2, the average number of steps between sampling instants for these simulations are compared with the performance measure

$$J_{\text{perf}} = \sum_{k=0}^{T_{\text{sim}}-1} \min_{\substack{y_k \in \mathcal{Y} \\ v_k \in K\mathcal{Y}}} \ell(x_k - y_k, u_k - v_k) \qquad (17)$$

for different values of β. Additionally, the performance bound $\beta\bar{V}_N^1(x_0)$ is given for comparison. While this performance bound is clearly conservative, from the comparison in the table it can be concluded that a lot of communication between plant and controller can be saved without much loss of performance. In Figure 13, several closed-loop trajectories are depicted for $\beta = 1.75$. It is important to note that the possible non-uniqueness of the minimizing input in (11) may influence the closed-loop behavior. That is, for the same disturbance sequence, the average performance as well as the average sampling frequency might be different for different minimizing inputs.

Remark 7. YALMIP (Löfberg (2004)), IBM ILOG CPLEX Optimization Studio (IBM (2014)), and the Multi-Parametric Toolbox 3.0 (Herceg et al. (2013)) were used in the simulations. In many cases, the solver reported "numerical problems" when solving the MPC problems (while still producing an output).

5. CONCLUSIONS AND OUTLOOK

We have presented a robust self-triggered model predictive controller based on Tube MPC methods. As shown in the numerical examples, the algorithm allows a trade-off between the average sampling rate and the guaranteed asymptotic bound on the system state. Future research directions include the output feedback case, further reducing the average sampling rate while guaranteeing the same asymptotic bound, and reducing the computational complexity.

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